The Illumination Conjecture for Spindle Convex Bodies *

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Abstract

A subset of the d-dimensional Euclidean space having nonempty interior is called a *spindle convex body* if it is the intersection of (finitely or infinitely many) congruent d-dimensional closed balls. The spindle convex body is called a "fat" one, if it contains the centers of its generating balls. The main result of this paper is a proof of the Illumination Conjecture for "fat" spindle convex bodies in dimensions greater than or equal to 15.

1 Introduction

Let $\mathbf{K}$ be a convex body (i.e., a compact convex set with nonempty interior) in the d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. According to Boltyanski [5] the direction $\mathbf{v} \in \mathbb{S}^{d-1}$ (i.e., the unit vector $\mathbf{v}$ of $\mathbb{E}^d$) illuminates the boundary point $\mathbf{b}$ of $\mathbf{K}$ if the halfline emanating from $\mathbf{b}$ having direction vector $\mathbf{v}$ intersects the interior of $\mathbf{K}$, where $\mathbb{S}^{d-1} \subset \mathbb{E}^d$ denotes the $(d-1)$-dimensional unit sphere centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$. Furthermore, the directions $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ illuminate $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Finally, the smallest n for which there exist n directions that illuminate $\mathbf{K}$ is called the *illumination number* of $\mathbf{K}$ denoted by $I(\mathbf{K})$. An equivalent but somewhat different looking concept

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of illumination was introduced by Hadwiger in [10]. There he proposed to use point sources instead of directions for the illumination of convex bodies. Based on these circumstances the following conjecture, that was independently raised by Boltyanski [5] and Hadwiger [10] in 1960, is called the *Boltyanski–Hadwiger Illumination Conjecture*: The illumination number $I(\mathbf{K})$ of any convex body $\mathbf{K}$ in $\mathbb{E}^d$, is at most 2^d and $I(\mathbf{K}) = 2^d$ if and only if $\mathbf{K}$ is an affine d-cube.

Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$ and let F be a face of $\mathbf{K}$ (i.e., let F be the intersection of a supporting hyperplane of $\mathbf{K}$ with the boundary of $\mathbf{K}$). Recall that the *Gauss image* $\nu(F)$ of the face F is the set of all points (i.e. unit vectors) $\mathbf{u} \in \mathbb{S}^{d-1} \subset \mathbb{E}^d$ with the property that the supporting hyperplane of $\mathbf{K}$ with outer normal vector $\mathbf{u}$ contains F. It is easy to see that the Gauss images of distinct faces of $\mathbf{K}$ have disjoint relative interiors in $\mathbb{S}^{d-1}$ and $\nu(F)$ is compact and spherically convex for any face F. (Recall that a set $Y \subset \mathbb{S}^{d-1}$ is spherically convex if it is contained in an open hemisphere of $\mathbb{S}^{d-1}$ and for every $\mathbf{y}_1, \mathbf{y}_2 \in Y$ the shorter great-circular arc of $\mathbb{S}^{d-1}$ connecting $\mathbf{y}_1$ with $\mathbf{y}_2$ is in Y.) Now, let $Y \subset \mathbb{S}^{d-1}$ be a set of finitely many points. Then the *covering radius* of Y is the smallest positive real number r with the property that the family of $(d-1)$-dimensional closed spherical balls of (angular) radii r centered at the points of Y cover $\mathbb{S}^{d-1}$. The following, rather basic principle, seems to be new and can be quite useful for estimating the illumination numbers of some convex bodies in particular, in low dimensions.

Lemma 1.1 *Let $\mathbf{K} \subset \mathbb{E}^d$, $d \geq 3$, be a convex body and let r be a positive real number with the property that the Gauss image $\nu(F)$ of any face F of $\mathbf{K}$ can be covered by a $(d-1)$-dimensional closed spherical ball of (angular) radius r in $\mathbb{S}^{d-1}$. Moreover, assume that there exist k points of $\mathbb{S}^{d-1}$ with covering radius R satisfying the inequality $r + R \leq \frac{\pi}{2}$. Then $I(\mathbf{K}) \leq k$.*

In what follows we are going to study sets called *spindle convex bodies*. Based on the recent paper [3] of the author, Lángi, Naszódi and Papez we can introduce them as follows. A subset of $\mathbb{E}^d$ having nonempty interior is called a *spindle convex body* if it is the intersection of (finitely or infinitely many) congruent d-dimensional closed balls. Here without loss of generality we assume that the congruent balls generating our spindle convex bodies are all of unit radii. Also, it is convenient to use the notation $\mathbf{B}[X]$ for the spindle convex body that is the intersection of the closed d-dimensional unit balls centered at the points of the compact set $X \subset \mathbb{E}^d$. For a comprehensive list of properties of spindle convex bodies we refer the interested reader to [3].

Now, let us take the spindle convex body $\mathbf{B}[X]$ in $\mathbb{E}^3$. First, observe that if the Euclidean diameter $\mathrm{diam}(X)$ of X satisfies the inequality $\mathrm{diam}(X) \leq 0.577$ (resp., $\mathrm{diam}(X) \leq 0.774$), then for the spherical diameter $\mathrm{Sdiam}(\nu(F))$ of the Gauss image $\nu(F)$ of an arbitrary face F of $\mathbf{B}[X]$ the inequality

$$\mathrm{Sdiam}(\nu(F)) \leq 2\arcsin(\frac{0.577}{2}) < 33.5364^\circ$$

$$(\text{resp., } \mathrm{Sdiam}(\nu(F)) \leq 2\arcsin(\frac{0.774}{2}) < 45.5360^\circ)$$

holds. (We note that for the purpose of this discussion we use the degree measure for angles following [9].) Thus, using the spherical Jung theorem [8], we obtain that the Gauss image $\nu(F)$ of any face F of $\mathbf{B}[X]$ can be covered by a 2-dimensional closed spherical disk of (angular) radius $\leq \arcsin\frac{0.577}{\sqrt{3}} < 19.459^\circ$ (resp., $\leq \arcsin\frac{0.774}{\sqrt{3}} < 26.543^\circ$). Second, recall the well-known spherical codes (see [9]) according to which on $\mathbb{S}^2$ there are 4 (resp., 5) points with covering radius $< 70.529^\circ$ (resp., $< 63.435^\circ$). Hence, Lemma 1.1 implies the following statement.

Corollary 1.2 *Let $\mathbf{B}[X]$ be a spindle convex body in $\mathbb{E}^3$.*

(i) If $0 < \mathrm{diam}(X) \leq 0.577$, then $I(\mathbf{B}[X]) = 4$;

(ii) If $0.577 < \mathrm{diam}(X) \leq 0.774$, then $I(\mathbf{B}[X]) \leq 5$.

The related statement that if $0 < \mathrm{diam}(X) \leq 1$, then $I(\mathbf{B}[X]) \leq 6$ has already been proved in [3]. Clearly, Corollary 1.2 suggests to attack the Illumination Conjecture for spindle convex bodies in $\mathbb{E}^3$ by letting $0 < \mathrm{diam}(X) < 2$ to get arbitrarily close to 2 while satisfying $0 < \mathrm{cr}(X) < 1$, where $\mathrm{cr}(X)$ denotes the radius of the unique smallest 3-dimensional closed ball containing X. In connection with this, it is natural to expect that the illumination number of any spindle convex body in $\mathbb{E}^3$ is always strictly less than 8.

Also, it is natural to expect that estimates similar to Corollary 1.2 exist in higher dimensions. For more details on that we refer the interested reader to the recent paper [4] of the author and Kiss. However, the following approach based on Schramm's paper [14] is more efficient if the dimension is sufficiently large. Namely, in [14] Schramm gave a proof of the Illumination Conjecture for all convex bodies of constant width in dimension greater than or equal to 16. In fact, he has proved the following inequality. If $\mathbf{W}$ is an arbitrary convex body of constant width in $\mathbb{E}^d, d \geq 3$, then

$$I(\mathbf{W}) < 5d\sqrt{d}(4 + \ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}.$$

By taking a closer look of the proof of the above upper bound of Schramm published in [14], and making the necessary modifications it turnes out that the estimate in question can be somewhat improved, but more importantly it can be extended to the family of "fat" spindle convex bodies, which is much larger than the family of convex bodies of constant width. Thus, we have the following theorem.

Theorem 1.3 *Let* $\mathbf{B}[X]$ *be an arbitrary spindle convex body in* $\mathbb{E}^d$*,* $d \geq 3$*, with* $\mathrm{diam}(X) \leq 1$*. Then*

$$I(\mathbf{B}[X]) < 4\left(\frac{\pi}{3}\right)^{\frac{1}{2}} d^{\frac{3}{2}}(3+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}} < 5d^{\frac{3}{2}}(4+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}.$$

On the one hand, $4\left(\frac{\pi}{3}\right)^{\frac{1}{2}} d^{\frac{3}{2}}(3+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}} < 2^d$ for all $d \geq 15$. (Moreover, for every $\epsilon > 0$ if d is sufficiently large, then $I(\mathbf{B}[X]) < \left(\sqrt{1.5}+\epsilon\right)^d = (1.224\ldots+\epsilon)^d$.) On the other hand, based on the elegant construction of Kahn and Kalai [11], it is known (see [1]), that if d is sufficiently large, then there exists a finite subset X'' of $\{0,1\}^d$ in $\mathbb{E}^d$ such that any partition of X'' into parts of smaller diameter requires more than $(1.2)^{\sqrt{d}}$ parts. Let X' be the (positive) homothetic copy of X'' having unit diameter and let X be the (not necessarily unique) convex body of constant width one containing X'. Then it follows via standard arguments that $I(\mathbf{B}[X]) > (1.2)^{\sqrt{d}}$ with $X = \mathbf{B}[X]$.

2 Proof of Lemma 1.1

Recall the following well-known observation on illumination. For the convenience of the reader and for notational reasons we include its short proof. (For more information on different approaches to illumination we refer the interested reader to [2] and the relevant references listed there.) Let the open ball centered at the point $\mathbf{p} \in \mathbb{S}^{d-1}$ having (angular) radius $0 < \alpha < \pi$ in the spherical space $\mathbb{S}^{d-1}$ be denoted by $C(\mathbf{p}, \alpha)$ and let us call it the *open spherical cap* of $\mathbb{S}^{d-1}$ with center $\mathbf{p}$ and radius α. In particular, $C(\mathbf{p}, \frac{\pi}{2})$ will be called the *open hemisphere* of $\mathbb{S}^{d-1}$ with center $\mathbf{p}$.

Lemma 2.1 *Let* $\mathbf{K}$ *be a convex body in* $\mathbb{E}^d$*,* $d \geq 3$*, and let* $\mathbf{b} \in \mathrm{bd}(\mathbf{K})$ *be an arbitrary boundary point of* $\mathbf{K}$*. Moreover, let* $F_{\mathbf{b}}$ *denote the smallest dimensional face of* $\mathbf{K}$ *containing* $\mathbf{b}$*. Then* $\mathbf{b}$ *is illuminated by the direction*

$\mathbf{v} \in \mathbb{S}^{d-1}$ *if and only if*

$$\nu\left(F_{\mathbf{b}}\right) \subset C\left(-\mathbf{v}, \frac{\pi}{2}\right).$$

Furthermore, $I(\mathbf{K})$ is the smallest number of open hemispheres of $\mathbb{S}^{d-1}$ with the property that the Gauss image of each face of $\mathbf{K}$ is contained in at least one of the given open hemispheres.

Proof: It is convenient to use the following notation. For a set $A \subset \mathbb{S}^{d-1}$ let $A^{+} := \{\mathbf{x} \in \mathbb{S}^{d-1} \mid \langle \mathbf{x}, \mathbf{y} \rangle > 0 \text{ for all } \mathbf{y} \in A\}$.

First, observe that the halfline emanating from $\mathbf{b} \in \operatorname{relint}(F_{\mathbf{b}})$ (with $\operatorname{relint}(\cdot)$ standing for the relative interior of the corresponding set) having direction vector $\mathbf{v}$ intersects the interior of $\mathbf{K}$ if and only if $-\mathbf{v} \in \nu\left(F_{\mathbf{b}}\right)^{+}$. Second, observe that $-\mathbf{v} \in \nu\left(F_{\mathbf{b}}\right)^{+}$ if and only if $\nu\left(F_{\mathbf{b}}\right) \subset C\left(-\mathbf{v}, \frac{\pi}{2}\right)$. This completes the proof of Lemma 2.1. □

Now, we turn to the proof of Lemma 1.1. Let $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ be the family of points in $\mathbb{S}^{d-1}$ with covering radius R. Moreover, let $B_i \subset \mathbb{S}^{d-1}$ be the $(d-1)$-dimensional closed spherical ball of radius R centered at the point $\mathbf{p}_i$ in $\mathbb{S}^{d-1}$, $1 \leq i \leq k$. Finally, let C_i be the open hemisphere of $\mathbb{S}^{d-1}$ with center $\mathbf{p}_i$, $1 \leq i \leq k$. Based on Lemma 2.1 it is sufficient to show that the Gauss image of each face of $\mathbf{K}$ is contained in at least one of the open hemispheres $C_i, 1 \leq i \leq k$.

Now, let F be an arbitrary face of the convex body $\mathbf{K} \subset \mathbb{E}^d$, $d \geq 3$, and let B_F denote the smallest $(d-1)$-dimensional closed spherical ball of $\mathbb{S}^{d-1}$ with center $\mathbf{f} \in \mathbb{S}^{d-1}$ which contains the Gauss image $\nu(F)$ of F. By assumption the radius of B_F is at most r. As the family $\{B_i, 1 \leq i \leq k\}$ of balls forms a covering of $\mathbb{S}^{d-1}$ therefore $\mathbf{f} \in B_j$ for some $1 \leq j \leq k$. If in addition, we have that $\mathbf{f} \in \operatorname{Sint}(B_j)$ (where $\operatorname{Sint}(\cdot)$ denotes the (spherical) interior of the corresponding set in $\mathbb{S}^{d-1}$), then the inequality $r + R \leq \frac{\pi}{2}$ implies that $\nu(F) \subset C_j$. If $\mathbf{f}$ does not belong to the interior of any of the sets $B_i, 1 \leq i \leq k$, then clearly $\mathbf{f}$ must be a boundary point of at least d sets of the family $\{B_i, 1 \leq i \leq k\}$. Then either we find a C_i containing $\nu(F)$ or we end up with d members of the family $\{C_i, 1 \leq i \leq k\}$ each being tangent to B_F at some point of $\nu(F)$. Clearly, the later case can occur only for finitely many $\nu(F)$'s and so, by taking a proper congruent copy of the open hemispheres $\{C_i, 1 \leq i \leq k\}$ within $\mathbb{S}^{d-1}$ (under which we mean to avoid finitely many so-called prohibited positions) we get that each $\nu(F)$ is contained in at least one member of the family $\{C_i, 1 \leq i \leq k\}$. This completes the proof of Lemma 1.1. □

3 Proof of Theorem 1.3

3.1 On the boundary of spindle convex hulls

Let $X \subset \mathbb{E}^d$, $d \geq 3$, be a compact set with $\mathrm{cr}(X) < 1$, where $\mathrm{cr}(X)$ denotes the radius of the smallest d-dimensional closed Euclidean ball containing X. For the following investigations it will be more proper to use the normal images than the Gauss images of the boundary points of $\mathbf{B}[X]$. The *normal image* $N_{\mathbf{B}[X]}(\mathbf{b})$ of an arbitrary boundary point $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$ of $\mathbf{B}[X]$ is defined as

$$N_{\mathbf{B}[X]}(\mathbf{b}) := -\nu(\{\mathbf{b}\})$$

In other words, $N_{\mathbf{B}[X]}(\mathbf{b}) \subset \mathbb{S}^{d-1}$ is the set of inward unit normal vectors of all hyperplanes that support $\mathbf{B}[X]$ at $\mathbf{b}$. Clearly, $N_{\mathbf{B}[X]}(\mathbf{b})$ is a closed spherically convex subset of $\mathbb{S}^{d-1}$. Moreover, Lemma 2.1 implies in a straighforward way that the direction $\mathbf{u} \in \mathbb{S}^{d-1}$ illuminates the boundary point $\mathbf{b}$ of the convex body $\mathbf{B}[X]$ if and only if $\mathbf{u} \in N_{\mathbf{B}[X]}(\mathbf{b})^{+}$.

We will need the following definitions and lemma from [3]. Let $\mathbf{a}$ and $\mathbf{b}$ be two points in $\mathbb{E}^d$. If $\|\mathbf{a}-\mathbf{b}\| < 2$, then the *(closed) spindle* of $\mathbf{a}$ and $\mathbf{b}$, denoted by $[\mathbf{a},\mathbf{b}]_s$, is defined as the union of circular arcs with endpoints $\mathbf{a}$ and $\mathbf{b}$ which have radii at least one and are shorter than a semicircle. If $\|\mathbf{a}-\mathbf{b}\| = 2$, then $[\mathbf{a},\mathbf{b}]_s := \mathbf{B}^d[\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}}, 1]$, where $\mathbf{B}^d[\mathbf{p}, r]$ denotes the (closed) d-dimensional ball centered at $\mathbf{p}$ with radius r in $\mathbb{E}^d$. If $\|\mathbf{a}-\mathbf{b}\| > 2$, then we define $[\mathbf{a},\mathbf{b}]_s$ to be $\mathbb{E}^d$. Next, a set $\mathbf{C} \subset \mathbb{E}^d$ is called *spindle convex* if, for any pair of points $\mathbf{a}, \mathbf{b} \in \mathbf{C}$, we have that $[\mathbf{a},\mathbf{b}]_s \subset \mathbf{C}$. Finally, let X be a set in $\mathbb{E}^d$. Then the *spindle convex hull* of X is the set defined by $\mathrm{conv}_s X := \bigcap\{C \subset \mathbb{E}^d | X \subset C \text{ and } C \text{ is spindle convex in } \mathbb{E}^d\}$. Also, recall that $S^{d-1}(\mathbf{c}, r) \subset \mathbb{E}^d$ denotes the $(d-1)$-dimensional sphere centered at $\mathbf{c}$ having radius r. A set $Y \subset S^{d-1}(\mathbf{c}, r)$ is *spherically convex* if it is contained in an open hemisphere of $S^{d-1}(\mathbf{c}, r)$ and for every $\mathbf{y}_1, \mathbf{y}_2 \in Y$ the shorter great-circular arc of $S^{d-1}(\mathbf{c}, r)$ connecting $\mathbf{y}_1$ with $\mathbf{y}_2$ is in Y. The *spherical convex hull* of a set $Y \subset S^{d-1}(\mathbf{c}, r)$ is defined in the natural way and it exists if, and only if, Y is in an open hemisphere of $S^{d-1}(\mathbf{c}, r)$. We denote it by $\mathrm{Sconv}(Y, S^{d-1}(\mathbf{c}, r))$. The following lemma proved in [3] describes some properties of the boundary of spindle convex hulls.

Lemma 3.1 *Let $X \subset \mathbb{E}^d$ be a compact set. If $\mathrm{cr}(X) < 1$ and $\mathbf{B}^d[\mathbf{q}, 1]$ is a closed unit ball containing X, then*

(i) $X \cap S^{d-1}(\mathbf{q}, 1)$ is contained in an open hemisphere of $S^{d-1}(\mathbf{q}, 1)$ and

(ii) $\mathrm{conv}_s(X) \cap S^{d-1}(\mathbf{q}, 1) = \mathrm{Sconv}(X \cap S^{d-1}(\mathbf{q}, 1), S^{d-1}(\mathbf{q}, 1))$.

Now, we are ready to prove the main lemma of this section.

Lemma 3.2 *Let $X \subset \mathbb{E}^d$, $d \geq 3$, be a compact set with* $\mathrm{cr}(X) < 1$. *Then the boundary of the spindle convex hull of X can be generated as follows:*

$$\mathrm{bd}\,(\mathrm{conv}_s(X)) = \bigcup_{\mathbf{b} \in \mathrm{bd}(\mathbf{B}[X])} \{\mathbf{b} + \mathbf{y} \mid \mathbf{y} \in N_{\mathbf{B}[X]}(\mathbf{b})\}.$$

Proof: Let $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$. Then (ii) of Lemma 3.1 implies that

$$\mathbf{b} + N_{\mathbf{B}[X]}(\mathbf{b}) = \mathrm{Sconv}(X \cap S^{d-1}(\mathbf{b}, 1), S^{d-1}(\mathbf{b}, 1)) = \mathrm{conv}_s(X) \cap S^{d-1}(\mathbf{b}, 1).$$

This together with the fact that

$$\bigcup_{\mathbf{b} \in \mathrm{bd}(\mathbf{B}[X])} N_{\mathbf{B}[X]}(\mathbf{b}) = \mathbb{S}^{d-1}$$

finishes the proof of Lemma 3.2. □

3.2 On the Euclidean diameter of spindle convex hulls and normal images

The following statement is essentially due to Meissner [13] (and it is proved in [6] as well).

Lemma 3.3 *If $X \subset \mathbb{E}^d$, $d \geq 3$, is a compact set with* $\mathrm{diam}(X) \leq 1$, *then*

$$\mathrm{diam}\,(\mathrm{conv}_s(X)) \leq 1.$$

For an arbitrary nonempty subset A of $\mathbb{S}^{d-1}$ let

$$U_{\mathbf{B}[X]}(A) := \left(\bigcup_{N_{\mathbf{B}[X]}(\mathbf{b}) \cap A \neq \emptyset} N_{\mathbf{B}[X]}(\mathbf{b}) \right) \subset \mathbb{S}^{d-1}.$$

Lemma 3.4 *Let $X \subset \mathbb{E}^d$, $d \geq 3$, be a compact set with* $\mathrm{diam}(X) \leq 1$ *and let $\emptyset \neq A \subset \mathbb{S}^{d-1}$ be given. Then*

$$\mathrm{diam}\left(U_{\mathbf{B}[X]}(A)\right) \leq 1 + \mathrm{diam}(A).$$

Proof: Let $\mathbf{y}_1 \in N_{\mathbf{B}[X]}(\mathbf{b}_1)$ and $\mathbf{y}_2 \in N_{\mathbf{B}[X]}(\mathbf{b}_2)$ be two arbitrary points of $U_{\mathbf{B}[X]}(A)$ with $\mathbf{b}_1, \mathbf{b}_2 \in \mathrm{bd}\,(\mathbf{B}[X])$. We need to show that $\|\mathbf{y}_1 - \mathbf{y}_2\| \leq 1 + \mathrm{diam}(A)$.

By Lemma 3.2 and by Lemma 3.3 we get that

$$\|(\mathbf{y}_1 - \mathbf{y}_2) + (\mathbf{b}_1 - \mathbf{b}_2)\| = \|(\mathbf{b}_1 + \mathbf{y}_1) - (\mathbf{b}_2 + \mathbf{y}_2)\| \leq 1.$$

Thus, the reverse triangle inequality yields that

$$\|\mathbf{y}_1 - \mathbf{y}_2\| \leq 1 + \|\mathbf{b}_2 - \mathbf{b}_1\|.$$

This means that in order to finish the proof of Lemma 3.4 it is sufficient to show that $\|\mathbf{b}_2 - \mathbf{b}_1\| \leq \mathrm{diam}(A)$. This can be done as follows. First, note that the sets $\mathbf{b}_1 + N_{\mathbf{B}[X]}(\mathbf{b}_1) \subset \mathrm{bd}\,(\mathrm{conv}_s(X))$ and $\mathbf{b}_2 + N_{\mathbf{B}[X]}(\mathbf{b}_2) \subset \mathrm{bd}\,(\mathrm{conv}_s(X))$ are separated by the hyperplane H of $\mathbb{E}^d$ that bisects the line segment connecting $\mathbf{b}_1$ to $\mathbf{b}_2$ and is perpendicular to it with $\mathbf{b}_1 + N_{\mathbf{B}[X]}(\mathbf{b}_1)$ (resp., $\mathbf{b}_2 + N_{\mathbf{B}[X]}(\mathbf{b}_2)$) lying on the same side of H as $\mathbf{b}_2$ (resp., $\mathbf{b}_1$). (All this follows in a direct way from the observation that a unit ball centered at an arbitrary point of $\mathrm{conv}_s(X)$ contains $\mathbf{B}[X]$.) Second, assume that $\|\mathbf{b}_2 - \mathbf{b}_1\| > \mathrm{diam}(A)$. Then this assumption together with the separating hyperplane H clearly imply that the Euclidean distance between the sets $N_{\mathbf{B}[X]}(\mathbf{b}_1)$ and $N_{\mathbf{B}[X]}(\mathbf{b}_2)$ is at least $\|\mathbf{b}_2 - \mathbf{b}_1\| > \mathrm{diam}(A)$, a contradiction (since by the assumption of Lemma 3.4 we have that $N_{\mathbf{B}[X]}(\mathbf{b}_1) \cap A \neq \emptyset$ and $N_{\mathbf{B}[X]}(\mathbf{b}_2) \cap A \neq \emptyset$). This completes the proof of Lemma 3.4. □

3.3 An upper bound for the illumination number

Let μ_{d-1} denote the standard probability measure on $\mathbb{S}^{d-1}$ and define

$$V_{d-1}(t) := \inf\{\mu_{d-1}(A^+) \mid A \subset \mathbb{S}^{d-1}, \mathrm{diam}(A) \leq t\},$$

where $0 < t \leq \sqrt{2}$. Moreover, let $n_{d-1}(\epsilon)$ denote the minimum number of closed spherical caps of $\mathbb{S}^{d-1}$ having Euclidean diameter ϵ such that they cover $\mathbb{S}^{d-1}$, where $0 < \epsilon \leq 2$.

Lemma 3.5

$$I(\mathbf{B}[X]) \leq 1 + \frac{\ln\left(n_{d-1}(\epsilon)\right)}{-\ln\left(1 - V_{d-1}(1+\epsilon)\right)}$$

holds for all $0 < \epsilon \leq \sqrt{2} - 1$ *and* $d \geq 3$.

Proof: Let $\emptyset \neq A \subset \mathbb{S}^{d-1}$ be given with $\mathrm{diam}(A) \leq 1+\epsilon \leq \sqrt{2}$. Then the spherical Jung theorem [8] implies that A is contained in a closed spherical cap of $\mathbb{S}^{d-1}$ having angular radius $0 < \arcsin\sqrt{\frac{d-1}{d}} < \frac{\pi}{2}$. Thus, A^+ contains a spherical cap of $\mathbb{S}^{d-1}$ having angular radius $\frac{\pi}{2} - \arcsin\sqrt{\frac{d-1}{d}} > 0$ and of course, A^+ is contained in an open hemisphere of $\mathbb{S}^{d-1}$. Hence, $0 < V_{d-1}(1+\epsilon) < \frac{1}{2}$ and so, the expression on the right in Lemma 3.5 is well-defined.

Let m be a positive integer satisfying

$$m > \frac{\ln\left(n_{d-1}(\epsilon)\right)}{-\ln\left(1 - V_{d-1}(1+\epsilon)\right)}.$$

It is sufficient to show that m directions can illuminate $\mathbf{B}[X]$. Let $n := n_{d-1}(\epsilon)$ and let $A_1, A_2, \ldots, A_n$ be closed spherical caps of $\mathbb{S}^{d-1}$ having Euclidean diameter ϵ and covering $\mathbb{S}^{d-1}$. By Lemma 3.4 we have

$$\mathrm{diam}\left(U_{\mathbf{B}[X]}(A_i)\right) \leq 1+\epsilon$$

for all $1 \leq i \leq n$ and therefore

$$\mu_{d-1}\left(U_{\mathbf{B}[X]}(A_i)^+\right) \geq V_{d-1}(1+\epsilon)$$

for all $1 \leq i \leq n$. Let the directions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be chosen at random, uniformly and independently distributed on $\mathbb{S}^{d-1}$. Thus, the probability that $\mathbf{u}_j$ lies in $U_{\mathbf{B}[X]}(A_i)^+$ is equal to $\mu_{d-1}\left(U_{\mathbf{B}[X]}(A_i)^+\right) \geq V_{d-1}(1+\epsilon)$. Therefore the probabilty that $U_{\mathbf{B}[X]}(A_i)^+$ contains none of the points $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is at most $(1 - V_{d-1}(1+\epsilon))^m$. Hence, the probability p that at least one $U_{\mathbf{B}[X]}(A_i)^+$ contains none of the points $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ satisfies

$$p \leq \sum_{i=1}^{n} (1 - V_{d-1}(1+\epsilon))^m < n\,(1 - V_{d-1}(1+\epsilon))^{\frac{\ln(n)}{-\ln\left(1-V_{d-1}(1+\epsilon)\right)}} = 1.$$

This shows that one can choose m directions say, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset \mathbb{S}^{d-1}$, such that each set $U_{\mathbf{B}[X]}(A_i)^+$, $1 \leq i \leq n$, contains at least one of them. We claim that the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ illuminate $\mathbf{B}[X]$. Indeed, let $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$. We will show that at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ illuminates the boundary point $\mathbf{b}$. As the spherical caps $A_1, A_2, \ldots, A_n$ form a covering of $\mathbb{S}^{d-1}$ therefore there exists an A_i with $A_i \cap N_{\mathbf{B}[X]}(\mathbf{b}) \neq \emptyset$. Thus, by definition $N_{\mathbf{B}[X]}(\mathbf{b}) \subset U_{\mathbf{B}[X]}(A_i)$ and therefore

$$N_{\mathbf{B}[X]}(\mathbf{b})^+ \supset U_{\mathbf{B}[X]}(A_i)^+.$$

$U_{\mathbf{B}[X]}(A_i)^+$ contains at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, say $\mathbf{v}_k$. Hence,

$$\mathbf{v}_k \in U_{\mathbf{B}[X]}(A_i)^+ \subset N_{\mathbf{B}[X]}(\mathbf{b})^+$$

and so, Lemma 2.1 yields that indeed, $\mathbf{v}_k$ illuminates the boundary point $\mathbf{b}$ of $\mathbf{B}[X]$, finishing the proof of Lemma 3.5. □

3.4 Schramm's lower bound for the proper measure of polars of sets of given diameter in spherical space

We need the following notation for the next statement. For $\mathbf{u} \in \mathbb{S}^{d-1}$ let $R_{\mathbf{u}} : \mathbb{E}^d \to \mathbb{E}^d$ denote the reflection about the line passing through the points $\mathbf{u}$ and $-\mathbf{u}$. Clearly, $R_{\mathbf{u}}(\mathbf{x}) = 2\langle \mathbf{x}, \mathbf{u}\rangle \mathbf{u} - \mathbf{x}$ for all $\mathbf{x} \in \mathbb{E}^d$. As the following two lemmas are taken from [14] with some minor changes in notation we quote them without proof.

Lemma 3.6 *Let $A \subset \mathbb{S}^{d-1}$ be a set of Euclidean diameter $0 < \mathrm{diam}(A) \leq t$ contained in the closed spherical cap $C[\mathbf{u}, \arccos a] \subset \mathbb{S}^{d-1}$ centered at $\mathbf{u} \in \mathbb{S}^{d-1}$ having angular radius $0 < \arccos a < \frac{\pi}{2}$ with $0 < a < 1$ and $0 < t \leq 2\sqrt{1-a^2}$. Then*

$$A^+ \cup R_{\mathbf{u}}(A^+) \supset C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right).$$

Lemma 3.7

$$V_{d-1}(t) \geq \frac{1}{\sqrt{8\pi d}}\left(\frac{3}{2} + \frac{\left(2-\frac{1}{d}\right)t^2 - 2}{4 - \left(2 - \frac{2}{d}\right)t^2}\right)^{-\frac{d-1}{2}}$$

for all $0 < t \leq \sqrt{2}$ and $d \geq 3$.

3.5 An upper bound for the number of sets of given diameter that are needed to cover spherical space

The following (simple) estimate is well-known (see for example [14]). We refer the interested reader for a proof to the proper section in [14].

Lemma 3.8

$$n_{d-1}(\epsilon) < \left(1 + \frac{4}{\epsilon}\right)^d$$

for all $0 < \epsilon \leq 2$ and $d \geq 3$.

3.6 The final upper bound for the illumination number

Now, we are ready for the proof of Theorem 1.3. As $x < -\ln(1-x)$ holds for all $0 < x < 1$, therefore by Lemma 3.5 we get that

$$I(\mathbf{B}[X]) \leq 1 + \frac{\ln\left(n_{d-1}(\epsilon)\right)}{-\ln\left(1 - V_{d-1}(1+\epsilon)\right)} < 1 + \frac{\ln\left(n_{d-1}(\epsilon)\right)}{V_{d-1}(1+\epsilon)}$$

holds for all $0 < \epsilon \leq \sqrt{2} - 1$ and $d \geq 3$. Now, let $\epsilon_0 = \sqrt{\frac{2d}{2d-1}} - 1$. As $0 < \epsilon_0 < \sqrt{2} - 1$ holds for all $d \geq 3$, therefore Lemma 3.7 and Lemma 3.8 together with the easy inequality $\epsilon_0 > \frac{4}{16d-1}$ yield that

$$I(\mathbf{B}[X]) < 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left(n_{d-1}(\epsilon_0)\right)$$

$$< 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left(\left(1 + \frac{4}{\epsilon_0}\right)^d\right) < 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left((16d)^d\right)$$

$$= 1 + 4\sqrt{\frac{\pi}{3}d}\sqrt{d}\left(\frac{3}{2}\right)^{\frac{d}{2}} (\ln 16 + \ln d) < 4\left(\frac{\pi}{3}\right)^{\frac{1}{2}} d^{\frac{3}{2}} (3 + \ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}},$$

finishing the proof of Theorem 1.3.

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