

SOLUTIONS to ASSIGNMENT 6.

MATH 311

FALL 2010

1. §9.1 #4(c). Let $T : \mathbf{P}_2 \rightarrow \mathbb{R}^2$ be given by $T(a+bx+cx^2) = (a+c, 2b)$. Consider the bases $B = \{1, x, x^2\}$ of \mathbf{P}_2 and $D = \{(1, 0), (1, -1)\}$ of \mathbb{R}^2 . Find the matrix of T corresponding to these bases, use it to compute $C_D[T(\mathbf{v})]$, and use that to compute $T(\mathbf{v})$ where $\mathbf{v} = a + bx + cx^2$. 8 marks

Solution. Compute $M_{DB}(T)$ directly:

$$M_{DB}(T) = [C_D[T(1)] \ C_D[T(x)] \ C_D[T(x^2)]] = [C_D[(1, 0)] \ C_D[(0, 2)] \ C_D[(1, 0)]] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Hence Theorem 2 §9.1 gives

$$C_D[T(\mathbf{v})] = M_{DB}(T)C_B(\mathbf{v}) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b+c \\ -2b \end{bmatrix}.$$

Finally we use this to recover the action of T :

$$T(\mathbf{v}) = (a + 2b + c)(1, 0) + (-2b)(1, -1) = (a + c, 2b)$$

as expected.

2. §9.1 #14. Let U be an invertible $n \times n$ matrix, and let $D = \{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ where \mathbf{f}_j is column j of U for each j . If B is the standard basis of \mathbb{R}^n , show that $M_{BD}(1_{\mathbb{R}^n}) = U$. 8 marks

Solution. Since B is the standard basis, we have $C_B(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$. Since B is the standard basis of \mathbb{R}^n , we have $C_B(X) = X$ for all $X \in \mathbb{R}^n$. Hence:

$$M_{BD}(1_{\mathbb{R}^n}) = [C_B(\mathbf{f}_1) \cdots C_B(\mathbf{f}_n)] = [\mathbf{f}_1 \cdots \mathbf{f}_n] = U.$$

3. §9.2 #15. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be any ordered basis of \mathbb{R}^n , written as columns. Define the matrix $Q = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ with the \mathbf{b}_i as its columns. Show that $QC_B(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. 8 marks

Solution. If $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$ where each $v_i \in \mathbb{R}$, then block multiplication gives

$$QC_B(\mathbf{v}) = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n = \mathbf{v}.$$

4. §9.3 #16(a). Let $V \xrightarrow{T} W \xrightarrow{S} V$ be linear transformations, and assume that $\dim(V)$ and $\dim(W)$ are finite. If $ST = 1_V$, show that $W = \text{im}(T) \oplus \text{ker}(S)$.

[Hint: If $\mathbf{w} \in W$, first show that $\mathbf{w} - TS(\mathbf{w}) \in \text{ker}(S)$.] 8 marks

Solution. As per the Hint: Since $ST = 1_V$, we have

$$S[\mathbf{w} - TS(\mathbf{w})] = S(\mathbf{w}) - ST(S\mathbf{w}) = S(\mathbf{w}) - S(\mathbf{w}) = 0.$$

Thus $\mathbf{w} - TS(\mathbf{w})$ lies in $\text{ker } S$ for all \mathbf{w} , so $\mathbf{w} = [\mathbf{w} - TS(\mathbf{w})] + T[S(\mathbf{w})]$ lies in $\text{ker } S + \text{im}(T)$. It follows that $W = \text{ker}(S) + \text{im}(T)$.

If \mathbf{w} is in $\text{ker}(S) \cap \text{im}(T)$, write $\mathbf{w} = T(\mathbf{v})$, \mathbf{v} in V . Then $\mathbf{v} = ST(\mathbf{v}) = S(\mathbf{w}) = \mathbf{0}$, so $\mathbf{w} = T(\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$. This means that $\text{ker}(S) \cap \text{im}(T) = \mathbf{0}$.

5. §9.3 #8(a). Define a linear transformation $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$ by

$$T(a + bx + cx^2) = (-a + 2b + c) + (a + 3b + c)x + (a + 4b)x^2.$$

If $U = \text{span}\{1, x + x^2\}$, show that U is T -invariant, find a block upper triangular matrix for T , and use that to compute the characteristic polynomial $c_T(x)$ of T . 8 marks

Solution. $T(1) = -1 + (x + x^2)$ is in U , and $T(x + x^2) = 3 + 4x + 4x^2 = 3 + 4(x + x^2)$ is in U . Hence U is T -invariant by Example 3. If $B = \{1, x + x^2, x^2\}$ then

$$M_B(T) = [C_D[T(1)] \ C_D[T(x)] \ C_D[T(x^2)]] = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence $c_T(x) = \det \begin{bmatrix} x+1 & -3 \\ -1 & x-4 \end{bmatrix} \det[x+1] = (x^2 - 3x - 7)(x+1)$.

Total: 40 marks