MATH 271 ASSIGNMENT 1 SOLUTIONS

1. For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation. You may use that every integer is either even or odd (but not both), but otherwise use only the definitions of even and odd integers.

   (a) $\forall a, b \in \mathbb{Z}$, if a is even and $a|b$ then b is even.

   (b) $\forall a, b \in \mathbb{Z}$, if a is even and $b|a$ then b is even.

   (c) $\forall a, b \in \mathbb{Z}$, if a is odd and $a|b$ then b is odd.

   (d) $\forall a, b \in \mathbb{Z}$, if a is odd and $b|a$ then b is odd.

   (a) This statement is **true**. Here is a proof.

   Let $a, b \in \mathbb{Z}$ so that a is even and $a|b$. This means that $a = 2k$ and $b = a\ell$ for some $k, \ell \in \mathbb{Z}$. Thus $b = a\ell = 2(k\ell)$ where $k\ell$ is an integer. So b is even by definition.

   (b) This statement is **false**. A counterexample is $a = 2$, $b = 1$. Then $a = 2$ is even and $b|a$ (since $1|2$), but $b = 1$ is not even.

   (c) This statement is **false**. A counterexample is $a = 1$, $b = 2$. Then $a = 1$ is odd and $a|b$ (since $1|2$), but $b = 2$ is not odd.

   (d) This statement is **true**. Here is a proof.

   Let $a, b \in \mathbb{Z}$ so that a is odd and $b|a$. We want to prove that b is odd. We will do this by contradiction. Suppose that b is not odd, which means we suppose that b is even. Then, since we have b even and $b|a$, we know from part (a) that a must be even. But this contradicts the assumption that a is odd. Thus by contradiction, b must be odd.

2. Prove or disprove each of the following:

   (a) $\forall a \in \mathbb{N}\ \exists b \in \mathbb{N}$ so that ab is composite.

   (b) $\exists a \in \mathbb{N}$ so that $\forall b \in \mathbb{N}$, ab is composite.

   (c) $\forall a \in \mathbb{N}\ \exists b \in \mathbb{N}$ so that $a + b$ is composite.

   (d) $\exists a \in \mathbb{N}$ so that $\forall b \in \mathbb{N}$, $a + b$ is composite.

   (a) This statement is **true**. Here is a proof.

   Let $a \in \mathbb{N}$ be arbitrary. We choose $b = 4$ regardless of the value of a. Then $ab = 4a = 2 \cdot 2a$, where both 2 and $2a$ are integers greater than 1. Therefore ab is composite.

   (b) This statement is **true**. Here is a proof.

   Choose $a = 4$. Then for any $b \in \mathbb{N}$, $ab = 4b = 2 \cdot 2b$, where both 2 and $2b$ are integers greater than 1. Therefore ab is composite.

   (c) This statement is **true**. Here is a proof.

   Let $a \in \mathbb{N}$ be arbitrary. We choose $b = 3a$ which is a positive integer. Then $a+b = 4a = 2 \cdot 2a$, where both 2 and $2a$ are integers greater than 1. Therefore $a + b$ is composite.

(d) This statement is **false**. We prove this by proving that its negation is true.

The negation of this statement is:

$$\forall a \in \mathbb{N}\ \exists b \in \mathbb{N} \text{ so that } a+b \text{ is not composite.}$$

Since $a+b \geq 2$ for all $a, b \in \mathbb{N}$, we can rewrite the negation as:

$$\forall a \in \mathbb{N}\ \exists b \in \mathbb{N} \text{ so that } a+b \text{ is prime.}$$

We want to prove this. Let $a \in \mathbb{N}$ be arbitrary. Since there are infinitely many primes (Theorem 3.7.4 on page 183), there must exist some prime $p > a$. Let $b = p - a$ which is a positive integer. Then $a + b = p$ which is prime, so the negation is true. Therefore the original statement is false.

3. For this question, do not use Exercises 13–16 on page 146 without proof.

   (a) Prove or disprove: $\forall x \in \mathbb{Q}$, if $x \in \mathbb{Z}$ then $x\lfloor x \rfloor \in \mathbb{Z}$.

   (b) Write out the converse of the statement in (a). Is it true? Give a proof or disproof.

   (c) Prove or disprove: $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$ then $x\lfloor x \rfloor \in \mathbb{Q}$.

   (d) Write out the converse of the statement in (c). Is it true? Give a proof or disproof.

(a) This statement is **true**. Here is a proof.

Let $x \in \mathbb{Z}$ be arbitrary. Then we know that $\lfloor x \rfloor = x$, so $x\lfloor x \rfloor = x^2 \in \mathbb{Z}$.

(b) The converse is: $\forall x \in \mathbb{Q}$, if $x\lfloor x \rfloor \in \mathbb{Z}$ then $x \in \mathbb{Z}$.

The converse is **false**. A counterexample is $x = 5/2$ which is a rational number. Then $x\lfloor x \rfloor = (5/2) \cdot \lfloor 5/2 \rfloor = (5/2) \cdot 2 = 5$ which is an integer, but $x = 5/2$ is not an integer.

(c) This statement is **true**. Here is a proof.

Let $x \in \mathbb{Q}$ be arbitrary. This means that $x = a/b$ for some integers a, b where $b \neq 0$. Also, $\lfloor x \rfloor \in \mathbb{Z}$. Thus $x\lfloor x \rfloor = (a/b)\lfloor x \rfloor = (a\lfloor x \rfloor)/b$ where $a\lfloor x \rfloor$ is an integer. Therefore $x\lfloor x \rfloor \in \mathbb{Q}$ by definition.

(d) The converse is: $\forall x \in \mathbb{R}$, if $x\lfloor x \rfloor \in \mathbb{Q}$ then $x \in \mathbb{Q}$.

The converse is **false**. A counterexample is $x = \sqrt{2}/2$. Since $0 < \sqrt{2}/2 < 1$, we get $\lfloor x \rfloor = \lfloor \sqrt{2}/2 \rfloor = 0$ and so $x\lfloor x \rfloor = 0$ which is a rational number. However we claim that $x = \sqrt{2}/2$ is not a rational number. To prove this we can use contradiction. Suppose that $\sqrt{2}/2$ is rational, which means that $\sqrt{2}/2 = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $\sqrt{2} = (2a)/b$ where $2a \in \mathbb{Z}$, so $\sqrt{2}$ would be rational, which is a contradiction (because of Theorem 3.7.1 on page 181).

MATH 271 ASSIGNMENT 2 SOLUTIONS

1. Let
$$S_n = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(2n-1)(2n)}\ .$$

(a) Find and simplify S_1, S_2 and S_3.

(b) Use part (a) (and more data if you need it) to guess a simple formula for S_n for any positive integer n.

(c) **Use mathematical induction** (or well ordering) to prove that your guess in part (b) is true for all positive integers n.

(a) We get

- $S_1 = \dfrac{1}{1\cdot 2} = \dfrac{1}{2}$,
- $S_2 = \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} = \dfrac{1}{6} + \dfrac{1}{12} = \dfrac{3}{12} = \dfrac{1}{4}$,
- $S_3 = \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \dfrac{1}{5\cdot 6} = \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} = \dfrac{5+3+2}{60} = \dfrac{10}{60} = \dfrac{1}{6}$.

(b) From part (a) we might guess that
$$S_n = \frac{1}{2n} \quad \text{for all integers } n \geq 1.$$

(c) *Basis step.* $S_n = 1/(2n)$ is true for $n = 1$ (and also for $n = 2$ and $n = 3$), by part (a).

Inductive step. Assume that $S_k = 1/(2k)$ for some integer $k \geq 1$. This means that
$$\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(2k-1)(2k)} = \frac{1}{2k}\ .$$

We want to prove that $S_{k+1} = 1/(2(k+1))$, which means we want to prove that
$$\frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots + \frac{1}{(2k+1)(2k+2)} = \frac{1}{2(k+1)}\ .$$

We notice that
$$\begin{aligned}
&\frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots + \frac{1}{(2k+1)(2k+2)} \\
&= \left[\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(2k-1)(2k)}\right] + \frac{1}{(2k)(2k+1)} \\
&\qquad + \frac{1}{(2k+1)(2k+2)} - \frac{1}{k(k+1)} \\
&= \frac{1}{2k} + \frac{1}{(2k)(2k+1)} + \frac{1}{(2k+1)(2k+2)} - \frac{1}{k(k+1)} \qquad \text{by assumption}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2k+1)(k+1)+(k+1)+k-2(2k+1)}{2k(2k+1)(k+1)} \\
&= \frac{2k^2+3k+1+k+1+k-4k-2}{2k(2k+1)(k+1)} \\
&= \frac{2k^2+k}{2k(2k+1)(k+1)} = \frac{1}{2(k+1)}\ ,
\end{aligned}
$$

so the statement is true for $n = k+1$. Therefore by induction, the statement is true for all integers $n \geq 1$.

Note. There is a nice short proof (not using induction) that $S_n = \frac{1}{2n}$ for all integers $n \geq 1$, using *telescoping*, which is a special technique mentioned in Example 4.1.10 on page 205 of the text. It makes use of the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for all $k > 0$. Then

$$
\begin{aligned}
S_n &= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(2n-1)(2n)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\
&= \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}\ . \qquad \text{(everything else cancels out)}
\end{aligned}
$$

2. The sequence $b_0, b_1, b_2, \ldots$ is defined by: $b_0 = 1$, $b_1 = 2$ and $b_n = 3b_{n-1} + b_{n-2}$ for all integers $n \geq 2$.

   (a) Calculate b_2, b_3 and b_4.

   (b) **Use mathematical induction** (or well ordering) to prove that $\gcd(b_{n+1}, b_n) = 1$ for all integers $n \geq 0$. [You may use Lemma 3.8.2 on page 193.]

   (c) **Use strong induction** (or well ordering) to prove that $b_n \leq 4^n$ for all integers $n \geq 0$.

   (a) We get

   $$b_2 = 3b_1 + b_0 = 3\cdot 2 + 1 = \mathbf{7}, \quad b_3 = 3b_2 + b_1 = 3\cdot 7 + 2 = \mathbf{23}, \quad b_4 = 3b_3 + b_2 = 3\cdot 23 + 7 = \mathbf{76}.$$

   (b) *Basis step.* When $n = 0$ we have

   $$\gcd(b_1, b_0) = \gcd(2, 1) = 1,$$

   so the statement is true for $n = 0$.

   Inductive step. Assume that $\gcd(b_{k+1}, b_k) = 1$ for some integer $k \geq 0$. We want to prove that $\gcd(b_{k+2}, b_{k+1}) = 1$. Since $b_{k+2} = 3b_{k+1} + b_k$, we know that

   $$
   \begin{aligned}
   \gcd(b_{k+2}, b_{k+1}) &= \gcd(b_{k+1}, b_k) \qquad \text{from Lemma 3.8.2} \\
   &= 1 \qquad \text{by assumption,}
   \end{aligned}
   $$

   so the statement is true for $n = k+1$. Therefore by induction, $\gcd(b_{n+1}, b_n) = 1$ for all integers $n \geq 0$.

(c) *Basis step.* When $n = 0$ the statement says $b_0 = 1 \leq 4^0$, which is true since $4^0 = 1$. When $n = 1$ the statement says $b_1 = 2 \leq 4^1$, which is also true since $4^1 = 4$.

Inductive step. Assume that $b_i \leq 4^i$ for all integers i satisfying $0 \leq i < k$, for some integer $k \geq 2$. We want to prove that $b_k \leq 4^k$. Since $k \geq 2$, we know by assumption that $b_{k-1} \leq 4^{k-1}$ and $b_{k-2} \leq 4^{k-2}$. Thus

$$\begin{aligned} b_k &= 3b_{k-1} + b_{k-2} \\ &\leq 3 \cdot 4^{k-1} + 4^{k-2} \qquad \text{by assumption} \\ &= 4^{k-2}(3 \cdot 4 + 1) = 4^{k-2} \cdot 13 < 4^{k-2} \cdot 4^2 = 4^k, \end{aligned}$$

so the statement is true for $n = k$. Therefore $b_n \leq 4^n$ for all integers $n \geq 0$, by strong induction.

3. You are given the following "while" loop:

[*Pre-condition*: m is a nonnegative integer, $a = 0$, $b = 1$, $i = 0$.]

while $(i \neq m)$

1. $a := 2b - a$
2. $b := 3a - 2b$
3. $i := i + 1$

end while

[*Post-condition*: $b - a = 2^m$.]

Loop invariant $I(n)$ is: $i = n, \quad a = 2^{n+1} - 2, \quad b = 2^{n+1} + 2^n - 2.$

(a) Prove the correctness of this loop with respect to the pre- and post-conditions.

(b) Suppose the "while" loop is as above, with the same pre-condition, except that statement 2 is replaced by: $b := 3a - 2b - 1$. Run through this new loop a few times to get data. Then find a post-condition that gives the final value of $b - a$, and an appropriate loop invariant, and prove the correctness of this new loop.

(a) We first need to check that the loop invariant holds when $n = 0$. But $I(0)$ says $i = 0$, $a = 2^1 - 2 = 0$, and $b = 2^1 + 2^0 - 2 = 1$, and these are all true by the pre-conditions.

So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$ where $k < m$. We want to prove that $I(k+1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that

$$i = k, \quad a = 2^{k+1} - 2, \quad b = 2^{k+1} + 2^k - 2,$$

and we now go through the loop.

- Step 1:

$$\begin{aligned} a := 2b - a &= 2(2^{k+1} + 2^k - 2) - (2^{k+1} - 2) \\ &= 2^{k+2} + 2^{k+1} - 4 - 2^{k+1} + 2 \\ &= 2^{k+2} - 2, \end{aligned}$$

which agrees with the formula for a in $I(k+1)$.

- Step 2:

$$\begin{aligned} b := 3a - 2b &= 3(2^{k+2} - 2) - 2(2^{k+1} + 2^k - 2) \\ &= 3 \cdot 2^{k+2} - 6 - 2^{k+2} - 2^{k+1} + 4 \\ &= 2^{k+2} + (2^{k+2} - 2^{k+1}) - 2 \\ &= 2^{k+2} + 2^{k+1} - 2, \end{aligned}$$

  which agrees with the formula for b in $I(k+1)$.
- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k+1)$.

Thus $I(k+1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $a = 2^{m+1} - 2$ and $b = 2^{m+1} + 2^m - 2$, and so $b - a = 2^m$ as required in the post-condition.

(b) If we set the variables to their pre-condition values of $a = 0$, $b = 1$ and $i = 0$, and run through the loop, the new values we get are

$$a = 2 \cdot 1 - 0 = 2, \quad b = 3 \cdot 2 - 2 \cdot 1 - 1 = 3, \quad i = 1.$$

If we continue to run through the loop, and keep track of the variables in a table, here is what we get:

n	0	1	2	3	4	5	6
a	0	2	4	6	8	10	12
b	1	3	5	7	9	11	13
i	0	1	2	3	4	5	6

It certainly looks like the post-condition should be $b - a = 1$ no matter what m is, and the loop invariant $I(n)$ should be: $i = n$, $a = 2n$, $b = 2n + 1$. From the pre-condition, $I(0)$ is true. So assume that $I(k)$ holds for some integer $k \geq 0$ where $k < m$, and we want to prove that $I(k+1)$ holds. So we are assuming that

$$i = k, \quad a = 2k, \quad b = 2k + 1,$$

and we now go through the loop.

- Step 1: $a := 2b - a = 2(2k+1) - 2k = 2k + 2 = 2(k+1)$,
  which agrees with the formula for a in $I(k+1)$.
- Step 2: $b := 3a - 2b - 1 = 3(2k+2) - 2(2k+1) - 1 = 2k + 3 = 2(k+1) + 1$,
  which agrees with the formula for b in $I(k+1)$.
- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k+1)$.

Thus $I(k+1)$ is true, as required.

Finally the loop stops when $i = m$, and then the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $a = 2m$ and $b = 2m + 1$, and so $b - a = 1$ as required in the post-condition.

MATH 271 ASSIGNMENT 3 SOLUTIONS

1. Prove or disprove each of the following statements. Proofs should use the "element" methods given in Section 5.2. [*Note*: $\mathcal{P}(X)$ denotes the power set of the set X.]

   (a) For all sets A, B, C, $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

   (b) For all sets A, B, C, $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

   (c) For all sets A, B, C, $A \times (B - C) = (A \times B) - (A \times C)$.

   (d) For all sets A and B, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   (e) For all sets A and B, $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   (f) For all sets A and B, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

   (a) This inequality is **true**. Here is a proof.

   Let A, B, C be arbitrary sets. Note that the left side of this inequality is a Cartesian product, which means that its elements will be ordered pairs. So let (a, b) be an arbitrary element of $A \times (B - C)$. This means that $a \in A$ and $b \in B - C$. Since $b \in B - C$, this means that $b \in B$ and $b \notin C$. Since $a \in A$ and $b \in B$, we get that $(a, b) \in A \times B$. But since $b \notin C$, we know that (a, b) cannot be an element of $A \times C$. Since $(a, b) \in A \times B$ but $(a, b) \notin A \times C$, we know $(a, b) \in (A \times B) - (A \times C)$. Therefore $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

   (b) Similarly, this inequality is **true**, and we can reverse our steps in part (a) to get a proof.

   Let (a, b) be an arbitrary element of $(A \times B) - (A \times C)$. This means that $(a, b) \in A \times B$ but $(a, b) \notin A \times C$. Since $(a, b) \in A \times B$, we know that $a \in A$ and $b \in B$. But since $(a, b) \notin A \times C$ although $a \in A$, we also know $b \notin C$. Thus $b \in B$ and $b \notin C$, which means $b \in B - C$. Thus $(a, b) \in A \times (B - C)$. Therefore $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

   (c) Since the inequalities in parts (a) and (b) both hold, we get that the equality in (c) holds for all sets A, B, C.

   (d) This inequality is **false**, and counterexamples are not hard to find. For example, let $A = \{1, 2\}$ and $B = \{1\}$. Then $\{1, 2\} \subseteq A$ and $\{1, 2\} \not\subseteq B$, so $\{1, 2\} \in \mathcal{P}(A)$ and $\{1, 2\} \notin \mathcal{P}(B)$, so $\{1, 2\} \in \mathcal{P}(A) - \mathcal{P}(B)$. However $A - B = \{2\}$, so $\{1, 2\} \notin \mathcal{P}(A - B)$. Therefore $\mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B)$.

   (e) This inequality is **false** no matter what sets we choose for A and B! To see this, let A and B be any sets. Notice that the empty set $\emptyset \subseteq A - B$ regardless of what A and B are, so $\emptyset \in \mathcal{P}(A - B)$. However, since $\emptyset \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(B)$, we get $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$. Therefore $\mathcal{P}(A - B) \not\subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   Note. You can prove that if X is any *nonempty* set so that $X \in \mathcal{P}(A - B)$, then $X \in \mathcal{P}(A) - \mathcal{P}(B)$. So the only counterexample to the inequality in part (e) is the empty set.

   (f) Since the inequality in (d) (or (e)) fails, the equality in (f) fails too.

2. (a) Prove that

$$n - \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil \quad \text{for all positive integers } n.$$

Here $\lceil x \rceil$ denotes the *ceiling* of the number x, as defined in §3.5. [*Hint*: do the cases n odd and n even separately.]

(b) The sequence $A_0, A_1, A_2, \ldots$ of sets is defined as follows:

$$A_0 = \emptyset, \quad \text{and } A_n = \{1, 2, \ldots, n\} - A_{n-1} \text{ for all integers } n \geq 1.$$

Find A_1, A_2 and A_3.

(c) For the sets A_n defined in part (b), prove **by induction on** n that $N(A_n) = \lceil n/2 \rceil$ for every integer $n \geq 0$. [*Hint*: Theorem 6.3.2 on page 322. $N(X)$ denotes the number of elements in the set X.]

(a) *Case* 1: Assume n is odd. Then $n = 2k+1$ for some integer k. So

$$\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{2k}{2} \right\rceil = k$$

and

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = \left\lceil k + \frac{1}{2} \right\rceil = k+1,$$

so

$$n - \left\lceil \frac{n-1}{2} \right\rceil = (2k+1) - k = k+1 = \left\lceil \frac{n}{2} \right\rceil.$$

Case 2: Assume n is even. Then $n = 2k$ for some integer k. So

$$\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{2k-1}{2} \right\rceil = \left\lceil k - \frac{1}{2} \right\rceil = k$$

and

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k}{2} \right\rceil = k,$$

so

$$n - \left\lceil \frac{n-1}{2} \right\rceil = 2k - k = k = \left\lceil \frac{n}{2} \right\rceil.$$

(b) We get

$$A_1 = \{1\} - A_0 = \{1\} - \emptyset = \{1\},$$
$$A_2 = \{1,2\} - A_1 = \{1,2\} - \{1\} = \{2\},$$
$$A_3 = \{1,2,3\} - A_2 = \{1,2,3\} - \{2\} = \{1,3\}.$$

(c) *Basis step.* When $n = 0$, $\lceil n/2 \rceil = 0 = N(A_0)$ since $A_0 = \emptyset$.

Inductive step. Assume that $N(A_k) = \lceil k/2 \rceil$ for some integer $k \geq 0$. We want to prove that $N(A_{k+1}) = \lceil (k+1)/2 \rceil$. Note that $A_0 \subseteq \{1\}$ and (for $k > 0$)

$$A_k = \{1, 2, \ldots, k\} - A_{k-1} \subseteq \{1, 2, \ldots, k+1\}.$$

Thus

$$\begin{aligned} N(A_{k+1}) &= N(\{1, 2, \ldots, k+1\} - A_k) \quad \text{by recursion} \\ &= N(\{1, 2, \ldots, k+1\}) - N(A_k) \quad \text{by Theorem 6.3.2 on page 322} \\ &= (k+1) - \left\lceil \frac{k}{2} \right\rceil \quad \text{by assumption} \\ &= \left\lceil \frac{k+1}{2} \right\rceil \quad \text{by part (a) (using } n = k+1\text{)}, \end{aligned}$$

which is what we wanted to prove.

So by induction, $N(A_n) = \lceil n/2 \rceil$ for every integer $n \geq 0$.

3. A licence plate consists of any three letters (from the usual 26-letter alphabet) followed by any three digits. Find the number of licence plates with each of the following properties. You need not simplify your answers.

   (a) They contain exactly two different symbols.

   (b) They contain exactly three different symbols.

   (c) They contain at least two 8's, but no 4, and the letters HAL in some order.

   (d) They use three different letters in alphabetical order and three different digits in increasing order. [*Hint*: start by choosing the three letters.]

(a) For a licence plate to have exactly two different symbols, it must have only one kind of letter and only one kind of digit; for example AAA111 is such a licence plate. The number of choices for the letter is 26 and the number of choices for the digit is 10. So by the multiplication rule, the number of such licence plates is $26 \times 10 =$ **260**.

(b) For a licence plate to have exactly three different symbols, it must have either (i) one kind of letter and two kinds of digits (for example AAA112), or (ii) two kinds of letters and one kind of digit (for example AAB111). We count these two possibilities separately.

(i) The number of choices for the letter is 26. For the two digits, one of them (say x) will occur twice and the other (say y) only once. The number of ways to choose x is 10, and (no matter which digit we choose for x) the number of ways to then choose y is 9. There are three places for the single digit y to go, and then the two x's will have to go in the other two places reserved for the digits. So by the multiplication rule, the number of licence plates of type (i) is $26 \times 10 \times 9 \times 3 =$ **7020**.

(ii) We similarly count these licence plates. The number of choices for the digit is 10. For the two letters, one of them (say α) will occur twice and the other (say β) only once. The number of ways to choose α is 26, and (no matter which letter we choose for α) the number of ways to then choose β is 25. There are three places for the single letter β to go, and then the two α's will have to go in the other two places reserved for the letters. So by the multiplication rule, the number of licence plates of type (ii) is $10 \times 26 \times 25 \times 3 = \mathbf{19500}$.

Thus, by the addition rule, the total number of such licence plates is

$$7020 + 19500 = \mathbf{26520}.$$

(c) The number of ways to arrange the letters HAL in some order is $3! = 6$. If our third digit (besides the two 8's) is another 8, then there is only one way to arrange these three 8's, so the total number of such licence plates will be just **6**. On the other hand, if our third digit is not an 8, then there are 8 choices for it (any digit except 8 and 4). There are three places to put this third digit, so there are $6 \times 8 \times 3 = \mathbf{144}$ such licence plates this time. So by the addition rule, the total number of licence plates will be $6 + 144 = \mathbf{150}$.

(d) There are $\binom{26}{3}$ ways to choose three different letters, and only one way to arrange them, since they must be in alphabetical order. There are $\binom{10}{3}$ ways to choose three different digits, and only one way to arrange them, since they must be in increasing order. By the multiplication rule, the number of such licence plates is

$$\binom{26}{3}\binom{10}{3} = \frac{26 \times 25 \times 24}{3 \times 2} \cdot \frac{10 \times 9 \times 8}{3 \times 2} = 26 \times 25 \times 4 \times 10 \times 3 \times 4 = \mathbf{312000}.$$

1. Let n be a positive integer.

(a) Prove that $\displaystyle\sum_{i=1}^{n+1} i\binom{n+1}{i} = \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=2}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}$. [*Hint*: Pascal's Formula (page 360).]

(b) Use part (a) and induction on n to prove the identity $\displaystyle\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$ for all integers $n \geq 1$. [*Hint*: Replace $i-1$ by j in the last two sums in the formula in part (a). You may also use Example 6.7.2 on page 368.]

(c) Give a *combinatorial* proof for the identity in part (b). [*Hint*: From a group of n people, choose a committee of any size with one of the people in the committee designated as the chair. In how many ways can you do this?]

(d) Let $[n] = \{1, 2, 3, \ldots, n\}$. Use the identity in part (b) to prove that the number of functions $f : \mathcal{P}([n]) \to \mathcal{P}([n])$ satisfying $f(X) \subseteq X$ for all $X \in \mathcal{P}([n])$ is exactly $2^{n2^{n-1}}$. [*Hint*: How many choices do you have for $f(\emptyset)$? How many for $f(\{1\})$?]

(a) We get

$$\begin{aligned}
\sum_{i=1}^{n+1} i\binom{n+1}{i} &= \sum_{i=1}^{n+1} i\left[\binom{n}{i} + \binom{n}{i-1}\right] && \text{by Pascal's Formula (see (1) below)}\\
&= \sum_{i=1}^{n+1} i\binom{n}{i} + \sum_{i=1}^{n+1} i\binom{n}{i-1} && \text{by Theorem 4.1.1 part 1, page 207}\\
&= \sum_{i=1}^{n+1} i\binom{n}{i} + \sum_{i=1}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}\\
&= \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=2}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}. && \text{(see (2) below)}
\end{aligned}$$

Notes. (1) According to page 360 of the text, Pascal's Formula $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ is only valid if $i \leq n$. Thus, strictly speaking, it does not apply in the case $i = n+1$ in line 1 above. However, when $i = n+1$ Pascal's Formula says $\binom{n+1}{n+1} = \binom{n}{n+1} + \binom{n}{n}$, where $\binom{n+1}{n+1} = \binom{n}{n} = 1$, so it would still be true if we defined $\binom{n}{n+1} = 0$. But since $\binom{n}{n+1}$ would mean the number of $(n+1)$-element subsets of an n-element set, it should be set equal to 0, in which case Pascal's Formula is okay for $i = n+1$ as well.

(2) Since we can define $\binom{n}{n+1} = 0$, in the last line above we changed the upper limit of the first right-hand sum from $n+1$ to n without changing the sum. Similarly we changed the lower limit of the second right-hand sum from $i = 1$ to $i = 2$, because when $i = 1$ the term in the sum is $0\binom{n}{0} = 0$.

(b) *Basis step*: When $n = 1$, the identity says that $1\binom{1}{1} = 1 \cdot 2^0$ or $1 = 1$ which is true.

Inductive step: Assume that

$$\sum_{i=1}^{k} i\binom{k}{i} = k2^{k-1}$$

is true for some integer $k \geq 1$. We want to prove that

$$\sum_{i=1}^{k+1} i\binom{k+1}{i} = (k+1)2^k.$$

Putting $i - 1 = j$ as in the Hint, and changing the limits of summation accordingly, we get

$$\begin{aligned}
\sum_{i=1}^{k+1} i\binom{k+1}{i} &= \sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=2}^{k+1}(i-1)\binom{k}{i-1} + \sum_{i=1}^{k+1}\binom{k}{i-1} && \text{by part (a)} \\
&= \sum_{i=1}^{k} i\binom{k}{i} + \sum_{j=1}^{k} j\binom{k}{j} + \sum_{j=0}^{k}\binom{k}{j} && \text{by the Hint} \\
&= k2^{k-1} + k2^{k-1} + \sum_{j=0}^{k}\binom{k}{j} && \text{by the assumption} \\
&= 2k2^{k-1} + 2^k && \text{by Example 6.7.2} \\
&= k2^k + 2^k = (k+1)2^k,
\end{aligned}$$

which finishes the inductive step.

Therefore, by induction, the identity is true for all integers $n \geq 1$.

(c) Let's choose the committee and chair by first choosing the people on the committee and then choosing the chair from among them. For each integer i between 1 and n, there are $\binom{n}{i}$ ways to choose a committee of i people from the n people, and then i ways to choose one of these i committee members to be the chair. So by the multiplication rule, there are $i\binom{n}{i}$ ways to choose a committee of i members including a chair. Any size i from 1 to n is possible, so by the addition rule, there must be $\sum_{i=1}^{n} i\binom{n}{i}$ ways to choose the committee and chair this way.

Next we choose the committee and chair by first choosing the chair, then the rest of the committee. There are n ways to choose one person (the chair) from among all n people. Whomever we choose, there are $n-1$ people left who could be on the committee, and we can choose any subset of them to fill out the committee, so there are 2^{n-1} ways to choose the rest of the committee. By the multiplication rule, there are $n2^{n-1}$ ways to choose the committee and chair.

We have counted the same thing two different ways, so our answers must be equal. Thus the identity in part (b) must be true.

Note. For those of you who have taken calculus, here is another nice proof of the identity in part (b). By the Binomial Theorem,

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n.$$

Take the derivative of both sides of this identity: we get

$$\sum_{i=0}^{n} \binom{n}{i} i x^{i-1} = n(x+1)^{n-1}.$$

Putting $x = 1$ (and removing $i = 0$ from the above sum) gives us the identity in part (b).

(d) To define one such function $f : \mathcal{P}([n]) \to \mathcal{P}([n])$, we need to say what $f(X)$ is for each $X \in \mathcal{P}([n])$.

- $f(\emptyset)$ must be a subset of $\emptyset$, so there is only one choice, namely $f(\emptyset) = \emptyset$.
- $f(\{1\})$ must be a subset of $\{1\}$, so there are two choices, $\emptyset$ and $\{1\}$.
- Similarly, for each of the one-element subsets $X = \{x\}$ of $[n]$, there will be only two choices for $f(X)$, because X will have only two subsets.
- In general, if X is a k-element subset of $[n]$, then X will have 2^k subsets, so there will be 2^k choices for $f(X)$.

For each k, there are $\binom{n}{k}$ subsets of $[n]$ of size k, so by the multiplication rule the number of ways to define $f(X)$ for all subsets X of $[n]$ of size k is $2^k \cdot 2^k \cdot \ldots \cdot 2^k$ ($\binom{n}{k}$ factors) which is $2^{k\binom{n}{k}}$. Thus by the multiplication rule the number of such functions f is

$$1 \cdot 2^{\binom{n}{1}} \cdot 2^{2\binom{n}{2}} \cdot \ldots \cdot 2^{n\binom{n}{n}} = 2^{\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}} = 2^{\sum_{i=1}^{n} i\binom{n}{i}} = 2^{n2^{n-1}} \qquad \text{by part (b).}$$

2. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer n.

(a) Find all functions $f : [2] \to [2]$ such that $f(k) \le k \ \forall k \in [2]$.

(b) Find the number of functions $f : [n] \to [n]$ such that $f(k) \le k \ \forall k \in [n]$.

(c) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \le k \ \forall k \in [n]$.

(d) Find the number of functions $f : [n] \to [n]$ such that $f(k) \le k + 1 \ \forall k \in [n]$.

(e) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \le k + 1 \ \forall k \in [n]$.

(a) For any such function f, since $f(1) \le 1$ and $f(2) \le 2$ we need $f(1) = 1$ and $f(2) = 1$ or 2. Thus there are exactly **two** such functions (which we call f_1 and f_2). They are respectively defined by:

- $f_1(1) = 1$ and $f_1(2) = 1$;
- $f_2(1) = 1$ and $f_2(2) = 2$.

(b) Since, for every k, $f(k)$ must be one of the k values $1, 2, \ldots, k$, there is one choice for $f(1)$ (namely 1), two choices for $f(2)$ (namely 1 or 2), and so on up to n choices for $f(n)$ (namely any of $1, 2, \ldots, n$). Thus by the multiplication rule there are $1 \cdot 2 \cdot \ldots \cdot n = n!$ ways to assign all the values $f(1), f(2), \ldots, f(n)$, that is, **n!** different functions.

(c) If f must be one-to-one, then we still must assign $f(1) = 1$, but then we cannot assign $f(2)$ to be 1 too, so we must put $f(2) = 2$. Next we cannot let $f(3)$ be 1 or 2, so we must put $f(3) = 3$. Continuing in this way, we are forced to put $f(k) = k$ for each k, so there is just **one** one-to-one function $f : [n] \to [n]$, namely the identity function.

(d) Proceeding as in part (b), for each k, $f(k)$ must be one of the $k+1$ choices $1, 2, \ldots, k+1$, provided that $k < n$. So $f(1)$ can be 1 or 2, $f(2)$ can be 1, 2 or 3, and so on up to $f(n-1)$ which can be any of $1, 2, \ldots, n$. But $f(n)$ must still belong to $[n]$ so there are only n choices for $f(n)$. Thus by the multiplication rule the total number of functions is $2 \cdot 3 \cdot \ldots \cdot n \cdot n = \mathbf{n(n!)}$.

(e) Note that since $[n]$ is finite, a function $f : [n] \to [n]$ is onto if and only if it is one-to-one. So we are really just counting one-to-one functions again. Now $f(1)$ must be 1 or 2, so there are two choices for $f(1)$. Then $f(2)$ must be 1, 2 or 3, so removing whichever choice we made for $f(1)$ will leave two choices for $f(2)$. In general there will be $k+1$ choices for $f(k)$ (namely $1, 2, \ldots, k+1$), but after we remove the choices we make for $f(1), f(2), \ldots, f(k-1)$ we will always have exactly two choices left for $f(k)$. The exception again is that for $f(n)$ there are only n choices originally (namely $1, 2, \ldots, n$), and after we remove the choices we make for $f(1), f(2), \ldots, f(n-1)$ we will only have one choice left for $f(n)$. So in total there will be $2 \cdot 2 \cdot \ldots \cdot 2 \cdot 1 = \mathbf{2^{n-1}}$ onto functions.

3. Let $n \geq 3$ be an integer, and let $\{1, 2, \ldots, n\}$ be the vertices of the complete graph K_n.

(a) Find two different circuits of length 3 (that is, 3 edges) in K_4 which use the same three vertices. [*Note.* According to the definition of circuit on page 667, two circuits are different if they are not exactly the same sequence of vertices and edges.]

(b) Find the number of circuits of length 3 in K_n.

(c) Find the number of subgraphs of K_n which are connected and have exactly 3 vertices, all of degree 2.

(d) Find the number of Hamiltonian circuits for K_n.

(e) Find the number of subgraphs of K_n which are connected and have n vertices, all of degree 2.

(a) For example, circuits 1231 and 1321 both use the three vertices $1, 2, 3$, but they are different circuits because the vertices are in a different order.

(b) Notice that a circuit of length 3 in K_n must use three different vertices, because K_n has no loops, and so we cannot repeat any vertices. Since listing the vertices is enough to determine the circuit, we only need to count the number of sequences a, b, c, a of vertices of K_n, where a, b, c are different. There are n choices for the first (and last) vertex a, and for each choice of a there are $n-1$ choices for the second vertex b and then $n-2$ choices for the third vertex c. By the multiplication rule, the number of such circuits is $\mathbf{n(n-1)(n-2)}$.

(c) If a subgraph of K_n is connected and has exactly 3 vertices, all of degree 2, it means that the subgraph must look exactly like a circuit of length 3. So once we choose the three vertices we know the subgraph. The difference between this part and part (b) is that in this part, it only matters what the vertices are, not what order they are in. Therefore the number of such subgraphs is $\binom{\mathbf{n}}{\mathbf{3}}$.

(d) Similar to part (b), we need to find the number of sequences $a_1, a_2, \ldots, a_n, a_1$, where $a_1, a_2, \ldots, a_n$ are just the vertices $1, 2, \ldots, n$ in some order. So the answer is $\mathbf{n!}$.

(e) Similar to part (c), if a subgraph of K_n is connected and has n vertices, all of degree 2, then it must be a Hamiltonian circuit of K_n. But two different Hamiltonian circuits will give us the same subgraph if they have the same edges. So we need to count how many Hamiltonian circuits of K_n will have the same edges, and divide the number of Hamiltonian circuits by this factor to eliminate the duplicate subgraphs. For each Hamiltonian circuit, we could start at any of the n vertices and go around the circuit in either direction, and we would get different circuits but the same subgraph. So we need to divide the answer to part (d) by $2n$, getting $n!/(2n) = \mathbf{(n-1)!/2}$ different subgraphs.