MATHEMATICS 271 L01 WINTER 2010
QUIZ 4 SOLUTION

1. For each of the following questions, give a brief explanation on how you get the answer.

(a) How many positive four-digit integers can be formed using only the digits 1,2,3,4,5,6,7,8,9?
Solution: The answer is 9^4 because we have 9 choices for each of the four digits of the integers.

(b) How many integers **in part (a)** have the property that the digit 1 appears at least once?

Solution: The answer is $9^4 - 8^4$. There are a total of 9^4 integers in part (a), 8^4 of which do not have the digit 1 (now we have 8 choices for each of the four digits).

(c) How many integers **in part (b)** have the property that the sum of the digits is even?

Solution: Since the sum of the digits is even, the number of odd digits must be even, but we must have at least one digit to be 1, so the number of odd digits must be 2 or 4.

Case 1: The number of odd digits is 2. In this case we have $\binom{4}{2} \times 5^2 \times 4^2 - \binom{4}{2} \times 4^2 \times 4^2$ such integers. Here, the total number of four digit integers in which two digits are odd is $\binom{4}{2} \times 5^2 \times 4^2$ ($\binom{4}{2}$ ways of choosing places for odds, 5^2 ways of putting the oods in and 4^2 ways of putting the vens in) and similarly, the number of four digit integers in which there is no 1's is $\binom{4}{2} \times 4^2 \times 4^2$.

The answer for this part also is $\binom{4}{1}\binom{3}{1} \times 4 \times 4^2 + \binom{4}{2} \times 4^2$ which is the number of integers in part (b) which has exactly one 1 plus the number of integers in part (b) which has two 1's.

Case 2: The number of odd digits is 4, that is, all the digits are odd. In this case we have $5^4 - 4^4$ such integers. Here, the total number of four-odd-digit integers is 5^4, and the number of four-odd-digit integers in which all integers are not odd (are even) is 4^4.

Thus, the answer to this question is $\binom{4}{2} \times 5^2 \times 4^2 - \binom{4}{2} \times 4^2 \times 4^2 + 5^4 - 4^4$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{7x+6}{8}$ for each $x \in \mathbb{R}$. Is f onto? Prove your answer. ($\mathbb{R}$ is the set of all real numbers.)

Solution: f is onto and here is a proof. For any $y \in \mathbb{R}$, we choose $x = \dfrac{8y-6}{7}$. Then $x \in \mathbb{R}$ and

$$f(x) = \frac{7x+6}{8} = \frac{7\left(\frac{8y-6}{7}\right)+6}{8} = \frac{(8y-6)+6}{8} = \frac{8x}{8} = y.$$

Thus, f is onto

3. Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = \left\lfloor \dfrac{7x+6}{8} \right\rfloor$ for each $x \in \mathbb{Z}$. Is g one-to-one? Prove your answer. ($\mathbb{Z}$ is the set of all integers.)

Solution: g is not one-to-one. For example, $g(6) = \left\lfloor \frac{7 \times 6 + 6}{8} \right\rfloor = \left\lfloor \frac{48}{8} \right\rfloor = 6$ and $g(7) = \left\lfloor \frac{7 \times 7 + 6}{8} \right\rfloor = \left\lfloor \frac{55}{8} \right\rfloor = 6$. So $g(6) = g(7)$ but $6 \neq 7$ and hence g is not one-to-one.