MATHEMATICS 213 WINTER 2010
QUIZ 5 SOLUTION

1. Is the set $\{x,\ \sin^2 x,\ \cos^2 x\}$ linearly independent in the vector space $\mathbb{F}[0,\ 2\pi]$? Prove your answer.

Solution: This set is linearly independent. Suppose that for some $a, b, c \in \mathbb{R}$,

$$ax + b\sin^2 x + c\cos^2 x = 0 \qquad (1)$$

When $x = 0$, from (1) we get $c = 0$. Thus, (1) becomes

$$ax + b\sin^2 x = 0 \qquad (2)$$

Now, put $x = \pi$, from (2) we get $a\pi = 0$ and so $a = 0$. Thus, (2) becomes

$$b\sin^2 x = 0 \qquad (3)$$

Choose $x = \frac{\pi}{2}$, from (3) we get $b = 0$. Thus, the set $\{x,\ \sin^2 x,\ \cos^2 x\}$ is linearly independent in the vector space $\mathbb{F}[0,\ 2\pi]$

2. Find a basis of $\mathbb{R}^3$ that includes the vector $(1, -1, 1)$.

Solution: An example of such a base is $B = \{(1, -1, 1), (0, 1, 0), (0, 0, 1)\}$. Since $\dim \mathbb{R}^3 = 3$, any set of 3 independent vectors is a bsis. Thus, we only have to prove that B is linearly independent. Suppose that $a(1, -1, 1) + b(0, 1, 0) + c(0, 0, 1) = (0, 0, 0)$. Then we get, $a = 0$, $c - a = 0$ and $a + \in c = 0$. It follows that $a = b = c = 0$. Thus, B is a basis of $\mathbb{R}^3$.

3. Consider the linear transformation $T : \mathbb{M}_{22} \to \mathbb{M}_{22}$ defined by $T(A) = A^T + A$ for all $A \in \mathbb{M}_{22}$. Is T one-to-one? Prove your answer.

Solution: T is not one-to-one.We prove that $\ker T$ is not the zero subspace by showing there is $0 \neq A \in \mathbb{M}_{22}$ such that $T(A) = 0$. One of such A is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since $\ker T$ is not the zero subspace, T is not one-to-one.

4. Let $T : V \to W$ be a linear transformation, and let $\overrightarrow{v_1},\ \overrightarrow{v_2}, ..., \overrightarrow{v_k}$ be vectors in V. Prove that if $\{T(\overrightarrow{v_1}),\ T(\overrightarrow{v_2}), ..., T(\overrightarrow{v_k})\}$. is linearly independent then $\{\overrightarrow{v_1},\ \overrightarrow{v_2}, ..., \overrightarrow{v_k}\}$ is linearly independent.

Solution: Suppose that $\{T(\overrightarrow{v_1}),\ T(\overrightarrow{v_2}),\ ...,\ T(\overrightarrow{v_k})\}$ is linearly independent. We prove that $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent.

Suppose that $a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \ ... + a_k\overrightarrow{v_k} = \overrightarrow{0}$. Then since T is linear, we get $a_1T(\overrightarrow{v_1}) + a_2T(\overrightarrow{v_2}) + \ ... + a_kT(\overrightarrow{v_k}) = T(a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \ ... + a_k\overrightarrow{v_k}) = T\left(\overrightarrow{0}\right) = \overrightarrow{0}$. By the independence of $\{T(\overrightarrow{v_1}),\ T(\overrightarrow{v_2}),\ ...,\ T(\overrightarrow{v_k})\}$, we get $a_1 = a_2 = \ ... = a_k = 0$. Thus, $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent.