MATHEMATICS 213 L01 WINTER 2010
ASSIGNMENT 4 SOLUTION

Due at 12:00 on Monday, April 12, 2010. Students should try to do all problems. However, only one question will be marked for credit.

1. Consider the vector space V, and $\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w} \in V$. Prove or disprove each of the following statements:

(a) If$\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is linearly independent then $\{\overrightarrow{u}+\overrightarrow{v}+\overrightarrow{w}\}$ is linearly independent.
(b) The set $\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{u}+2\overrightarrow{v}\}$ is linearly dependent.
(c) $span\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\} = span\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-\overrightarrow{u}\}$.
(d)$\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is linearly independent if and only if $\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-\overrightarrow{u}\}$ is linearly independent.

Solution:

(a) This statement is true and here is a proof. Suppose that $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is linearly independent. We prove that $\{\overrightarrow{u}+\overrightarrow{v}+\overrightarrow{w}\}$ is linearly independent. Suppose that $a(\overrightarrow{u}+\overrightarrow{v}+\overrightarrow{w}) = \overrightarrow{0}$ for some $a \in \mathbb{R}$. Then $a\overrightarrow{u}+a\overrightarrow{v}+a\overrightarrow{w} = \overrightarrow{0}$ and so $a = 0$ by the independence of $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$.
(b) This statement is false. The set $\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{u}+2\overrightarrow{v}\}$ is not linearly dependent because $1(\overrightarrow{u}-\overrightarrow{v}) - 3(\overrightarrow{u}+\overrightarrow{v}) + 2\ (\overrightarrow{u}+2\overrightarrow{v}) = \overrightarrow{0}$.
(c) This statement is false, i.e., there exists a vector space V and vectors $\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w} \in V$ such that $span\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\} \neq span\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-\overrightarrow{u}\}$. For example, let $V = \mathbb{R}^3$ and choose $\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w} \in \mathbb{R}^3$ such that $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is linear ly independent. We note that $\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-\overrightarrow{u}\}$ is not linearly independent (because $1(\overrightarrow{u}-\overrightarrow{v}) + 1(\overrightarrow{v}-\overrightarrow{w}) + 1(\overrightarrow{w}-\overrightarrow{u}) = \overrightarrow{0}$). Hence, $\dim(span\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}) = 3$ and $\dim(span\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-\overrightarrow{u}\}) < 3$, and so $span\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\} \neq span\{\overrightarrow{u}-\overrightarrow{v},\ \overrightarrow{v}-\overrightarrow{w},\ \overrightarrow{w}-$
(d) This statement is false as seen in part (c)

2. Prove or disprove each of the following:
(a) There is a basis of $\mathbb{R}^3$ that includes the vector $(1,1,1)$.
(b) There is a basis of $\mathbb{P}_3$ consisting of polynomials whose coefficients sum to 4.
(c) There is a basis of $\mathbb{P}_3$ consisting of polynomials whose coefficients sum to 0.
(d)There is a basis of $\mathbb{M}_{22}$ consisting of matrices with the property that $A^2 = A$.

Solution:

(a) This statement is true. We prove that $B = \{(1,1,1), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$. In fact, since $\dim(\mathbb{R}^3) = 3$, we ony need to prove that B is linearly independent. Suppose that $a(1,1,1) + b(0,-1,-1) + c(0,0,1) = (0,0,0)$ for some $a,b,c \in \mathbb{R}$. Thus, $a = 0, a-b = 0$ and $a-b+c = 0$ which clearly implies that $a = b = c = 0$. Thus, B is linearly independent and so B is a basis of $\mathbb{R}^3$.
(b) This statement is true. In this case, such a basis is $\{4,\ 4x,\ 4x^2,\ 4x^3\}$.
(c) This statement is false. We prove that any set of 4 polynomials in $\mathbb{P}_3$ whose coefficients sum to 0 can not be a basis of $\mathbb{P}_3$. Let $B = \{a_0+a_1x+a_2x^2+a_3x^3,\ b_0+b_1x+b_2x^2+b_3x^3,\ c_0+c_1x+c_2x^2$ -
where $a_0+a_1+a_2+a_3 = b_0+b_1+b_2+b_3 = c_0+c_1+c_2+c_3 = d_0+d_1+d_2+d_3 = 0$. Then for any $p(x) \in spanB,\ p(x) = a(a_0+a_1x+a_2x^2+a_3x^3) + b(\ b_0+b_1x+b_2x^2+b_3x^3) +$

$c\,(\ c_0+c_1x+c_2x^2+c_3x^3)+d\,(\ d_0+d_1x+d_2x^2+d_3x^3)$ for some $a,b,c,d\in\mathbb{R}$. Then the sum of the coefficients of $p\,(x)$ is $a\,(a_0+a_1+a_2+a_3)+b\,(b_0+b_1+b_2+b_3)+c\,(c_0+c_1+c_2+c_3)+d\,(d_0+d_1+d_2+d_3)=0$.

Thus, we proved that if $p\,(x)\in spanB$ then the sum of the coefficients of $p\,(x)$ is 0. This implies that $1+x\notin spanB$, and so b is not a basis of $\mathbb{P}_3$.

(d) This statement is true. In this case, such a basis is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$.

3. Let $T:V\to W$ be a linear transformation. Prove or disprove each of the following

(a) If $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent then $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

(b) If $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent and T is one-to-one then $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

(c) If $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent then $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent.

(d) If $\dim(\ker T)\leq\dim W$ then $\dim W\geq\dfrac{1}{2}\dim V$.

Solution:

(a) This statement is false. For example, $V=W=\mathbb{R}$, and $T:\mathbb{R}\to\mathbb{R}$ defined by $T\,(x)=0$ for all $x\in\mathbb{R}$, that is, T is the zero operator. Now, the set $\{1\}$ is linearly independent in $\mathbb{R}$, but $\{T\,(1)\}=\{0\}$is not linearly independent in $\mathbb{R}$.

(b) This statement is true. Suppose that $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent and T is one-to-one. We prove that $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

Suppose that $a_1T\,(\overrightarrow{v_1})+a_2T\,(\overrightarrow{v_2})+\ ...+a_kT\,(\overrightarrow{v_k})=\overrightarrow{0}$. Then since T is linear, this is the same as $T\,(a_1\overrightarrow{v_1}+a_2\overrightarrow{v_2}+\ ...+a_k\overrightarrow{v_k})=\overrightarrow{0}=T\left(\overrightarrow{0}\right)$. This implies that $a_1\overrightarrow{v_1}+a_2\overrightarrow{v_2}+\ ...+a_k\overrightarrow{v_k}=\overrightarrow{0}$ (because T is one-to-one) and so by the independence of $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$, we get $a_1=a_2=...=a_k=0$. Thus, $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

(c) This statement is true. Suppose that $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$ is linearly independent and T is one-to-one. We prove that $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

Suppose that $a_1T\,(\overrightarrow{v_1})+a_2T\,(\overrightarrow{v_2})+\ ...+a_kT\,(\overrightarrow{v_k})=\overrightarrow{0}$. Then since T is linear, this is the same as $T\,(a_1\overrightarrow{v_1}+a_2\overrightarrow{v_2}+\ ...+a_k\overrightarrow{v_k})=\overrightarrow{0}=T\left(\overrightarrow{0}\right)$. This implies that $a_1\overrightarrow{v_1}+a_2\overrightarrow{v_2}+\ ...+a_k\overrightarrow{v_k}=\overrightarrow{0}$ (because T is one-to-one) and so by the independence of $\{\overrightarrow{v_1},\ \overrightarrow{v_2},\ ...,\ \overrightarrow{v_k}\}$, we get $a_1=a_2=\ ...=a_k=0$. Thus, $\{T\,(\overrightarrow{v_1})\,,\ T\,(\overrightarrow{v_2})\,,\ ...,\ T\,(\overrightarrow{v_k})\}$ is linearly independent.

(d) This statement is true. Suppose that $\dim(\ker T)\leq\dim W$. By the Dimension Theorem, $\dim V=\dim(ketT)+\dim(imT)\leq\dim W+\dim W=2\dim W$ and so $\dim W\geq\dfrac{1}{2}\dim V$.