MATHEMATICS 211 L03 B11/12 WINTER 2010
QUIZ 5 SOLUTION
Thursday, April 15, 2010

1. For the following, express your answer in the form $a + bi$.

(a) Compute $\left(1+\sqrt{3}i\right)^{-4}$.

Solution:

$$\begin{aligned} \left(1+\sqrt{3}i\right)^{-4} &= \left(2e^{i\frac{\pi}{3}}\right)^{-4} \\ &= 2^{-4}e^{i\left(-\frac{4\pi}{3}\right)} \\ &= \frac{1}{16}\left(\cos\left(-\frac{4\pi}{3}\right)+i\sin\left(-\frac{4\pi}{3}\right)\right) \\ &= \frac{1}{16}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= -\frac{1}{32}+\frac{\sqrt{3}}{32}i \end{aligned}$$

(b) Find all complex numbers z such that $z^3 = -27i$.

Solution: Let $z = re^{i\theta}$. The equation becomes $r^3e^{i3\theta} = 27e^{i(\pi+2k\pi)}$ where k is any integer.

Thus, $r^3 = 27$ and $3\theta = \pi + 2k\pi$ and hence, $r = 3$ and $\theta = \dfrac{\pi+2k\pi}{3}$. Therefore, we have $z = 3e^{i\left(\dfrac{\pi+2k\pi}{3}\right)}$

When $k = 0$, we have $z = 3e^{i\frac{\pi}{3}} = 3\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \frac{3}{2}+\frac{3\sqrt{3}}{2}i$

When $k = 1$, we have $z = 3e^{i\pi} = 3\left(-1+0i\right) = -3+0i$

When $k = 2$, we have $z = 3e^{i\frac{5\pi}{3}} = 3\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \frac{3}{2}-\frac{3\sqrt{3}}{2}i$

2. Consider the points $P\,(2,3,5)$ and $Q\,(8,-6,2)$.

(a) Find the coordinates of the two points M and N that trisect the line segment from P to Q.

Solution:

$$\begin{aligned} \overrightarrow{OM} &= \overrightarrow{OP}+\overrightarrow{PM} &&= \overrightarrow{OP}+\tfrac{1}{3}\overrightarrow{PQ} \\ &= \overrightarrow{OP}+\tfrac{1}{3}\left(\overrightarrow{OQ}-\overrightarrow{OP}\right) &&= \tfrac{2}{3}\overrightarrow{OP}+\tfrac{1}{3}\overrightarrow{OQ} \\ &= \tfrac{2}{3}\,[2,3,5]^T+\tfrac{1}{3}\,[8,-6,2]^T &&= [4,0,4]^T \end{aligned}$$

and,

$$\begin{aligned} \overrightarrow{ON} &= \overrightarrow{OP}+\overrightarrow{PN} &&= \overrightarrow{OP}+\tfrac{2}{3}\overrightarrow{PQ} \\ &= \overrightarrow{OP}+\tfrac{2}{3}\left(\overrightarrow{OQ}-\overrightarrow{OP}\right) &&= \tfrac{1}{3}\overrightarrow{OP}+\tfrac{2}{3}\overrightarrow{OQ} \\ &= \tfrac{1}{3}\,[2,3,5]^T+\tfrac{2}{3}\,[8,-6,2]^T &&= [6,-3,3]^T. \end{aligned}$$

Thus, the coordiates of the two points are $M\,(4,0,4)$ and $N\,(6,-3,3)$.

(b) Find an equation of the line that passes through the points P and Q.

Solution:

A point of the line is $P\,(2,3,5)$ and a direction of the line is $\frac{1}{3}\overrightarrow{PQ} = \frac{1}{3}\,[6,-3,-3]^T = [2,-1,-1]^T$. An equation of the line is

$[x,y,z]^T = [2,3,5]^T + t\,[2,-1,-1]^T$.

MATHEMATICS 211 L03 B9/10 WINTER 2010
QUIZ 5 SOLUTION
Wednesday, April 14, 2010

1. For each of the following, express your answer in the form $a+bi$.
(a) Compute $(1-i)^{10}$.
Solution:

$$\begin{aligned}(1-i)^{10} &= \left[\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}\right]^{10} &&= \left(\sqrt{2}\right)^{10} e^{i\left(-\frac{5\pi}{2}\right)} \\ &= 32\left(\cos\left(-\tfrac{5\pi}{2}\right)+i\sin\left(-\tfrac{5\pi}{2}\right)\right) &&= 32\,(0-i) \\ &= -32i\end{aligned}$$

(b) Find all complex numbers z such that $z^4 = 2\left(\sqrt{3}i-1\right)$.
Solution:

Let $z = re^{i\theta}$. The equation becomes $r^4e^{i4\theta} = 4e^{i\left(\frac{2\pi}{3}+2k\pi\right)}$ where k is any integer.

Thus, $r^4 = 4$ and $4\theta = \frac{2\pi}{3}+2k\pi$ and hence, $r=\sqrt{2}$ and $\theta = \dfrac{\pi}{6}+\dfrac{k\pi}{2}$. Therefore, we have $z = \sqrt{2}e^{i\left(\dfrac{\pi}{6}+\dfrac{k\pi}{2}\right)}$

$$\begin{aligned}
&\text{When } k=0, \text{ we have } & z &= \sqrt{2}e^{i\frac{\pi}{6}} &&= \sqrt{2}\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right) &&= \tfrac{\sqrt{6}}{2}+\tfrac{\sqrt{2}}{2}i \\
&\text{When } k=1, \text{ we have } & z &= \sqrt{2}e^{i\frac{2\pi}{3}} &&= \sqrt{2}\left(\tfrac{-1}{2}+\tfrac{\sqrt{3}}{2}i\right) &&= -\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{6}}{2}i \\
&\text{When } k=2, \text{ we have } & z &= \sqrt{2}e^{i\frac{7\pi}{6}} &&= \sqrt{2}\left(-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right) &&= -\tfrac{\sqrt{6}}{2}-\tfrac{\sqrt{2}}{2}i \\
&\text{When } k=3, \text{ we have } & z &= \sqrt{2}e^{i\frac{5\pi}{3}} &&= \sqrt{2}\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) &&= \tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{6}}{2}i
\end{aligned}$$

2. A parallelogram has sides AB, BC, CD and DA. Given $A\,(1,-1,2)$, $C\,(2,1,0)$ and the midpoint $M\,(1,0,-3)$ of AB. Find $\overrightarrow{BD}$.
Solution:

$$\begin{aligned}
\overrightarrow{BD} &= \overrightarrow{BA}+\overrightarrow{AC}+\overrightarrow{CD} \\
&= \overrightarrow{BA}+\overrightarrow{AC}+\overrightarrow{BA} && \text{because } \overrightarrow{CD}=\overrightarrow{BA} \\
&= \overrightarrow{AC}+2\overrightarrow{BA} \\
&= \overrightarrow{AC}+4\overrightarrow{MA} && \text{because } \overrightarrow{BA}=2\overrightarrow{MA} \\
&= \left(\overrightarrow{OC}-\overrightarrow{OA}\right)+4\left(\overrightarrow{OA}-\overrightarrow{OM}\right) \\
&= \overrightarrow{OC}+3\overrightarrow{OA}-4\overrightarrow{OM} \\
&= [2,1,0]^T+3\,[1,-1,2]^T-4\,[1,0,-3]^T \\
&= [1,-2,18]^T.
\end{aligned}$$

3. Find an equation of the plane which contains the points $A\,(1,2,1)$, $B\,(1,3,0)$ and $C\,(2,-1,3)$.
Solution: A point on the plane is $B\,(1,3,0)$, and a normal of the plane is $\overrightarrow{d} = -\overrightarrow{AB}\times\overrightarrow{AC}$ Now, we compute $\overrightarrow{d}$.

$$\begin{aligned}
\overrightarrow{d} &= -\overrightarrow{AB} \times \overrightarrow{AC} \\
&= -\left[0, 1, -1\right]^T \times \left[1, -3, 2\right]^T \\
&= -\left[\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix}, -\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}\right]^T \\
&= -\left[-1, -1, -1\right]^T \\
&= \left[1, 1, 1\right].
\end{aligned}$$

Thus, an equation of the plane is $x + y + z = 4$.