FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 221 (L 05/08)

FALL 2007 Time: 3 hours

[10] 1. Solve the system:

$$\begin{array}{ccccccccccc} x & + & 2y & + & 2z & + & u & + & w & = & 2 \\ 2x & + & 4y & + & 4z & + & u & + & 4w & = & 7 \end{array}$$

[10] 2. Let $A^{-1} = GFE$ where $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
Note that G, E and F are elementary matrices

(a) Find A.
(b) Find $\det A$.
(c) Solve the system $AX = B$ where $B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

[10] 3. Let A, B and C be 3×3 matrices so that $\det A = \det B$ and $\det C = 2$.

(a) Find $\det \left(2A^{-1}C^2B^T\right)^T$.
(b) Find $\det \left(C^{-1} + adj C\right)$.

[10] 4. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$.

(a) Find $adj A$.
(b) Compute $A \cdot adj A$.
(c) Find $\det A$ using part (b).

[10] 5. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Is A diagonalizable? If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

[10] 6. Let A and B be $n \times n$ matrices. Prove the following statements:

(a) If λ is an eigenvalue of A then $\lambda^2 - 2$ is an eigenvalue of $A^2 - 2I$.
(b) If $A = P^{-1}BP$ for some invertible matrix P then A and B have the same characteristic polynomial.

[10] 7. For the following, express your answers in the form $a + bi$ where a and b are real numbers.

(a) Compute $(1 - i)^{10}$.
(b) Find all complex numbers z so that $z^4 = -1$.

[10] 8. Consider the points $A\,(1, 1, 1)$, $B\,(1, 2, 3)$ and $C\,(2, 1, 3)$.

(a) Is the triangle with vertices A, B and C a right triangle? Explain.
(b) Find the area of the triangle with vertices A, B and C.
(c) Find an equation of the plane containing the points A, B and C.

[10] 9. Let P_1 be the plane with equation $x + y + z = 2$ and P_2 be the plane with equation $2x - y + z = 3$. Consider the point $A\,(3, 7, -2)$.

(a) Find an equation of the line that passes through the point A and is parallel to both of the planes P_1 and P_2.

(b) Find the shortest distance between the point A and the plane P_1 and find the coordinates of the point B on P_1 that is closest to A.

[10] 10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$.

(a) Find the matrix of T; that is, find a matrix A so that $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.

(b) Is T invertible? If T is invertible, find the matrix of T^{-1}.

(c) Is there a vector $\vec{a} \in \mathbb{R}^2$ so that $T\vec{a} = \left[\begin{smallmatrix}-3\\7\end{smallmatrix}\right]$? If so, find $\vec{a}$.