MATHEMATICS 211 L03 WINTER 2010 MIDTERM SOLUTION

1. Solve the system:

$$\begin{array}{ccccccccc} x & + & 2y & + & 2z & - & u & = & 2 \\ x & + & 2y & + & z & - & 3u & = & -1 \end{array}$$

Solution:

$$\left[\begin{array}{ccccc} 1 & 2 & 2 & -1 & 2 \\ 1 & 2 & 1 & -3 & -1 \end{array}\right] \; R_2 - R_1 \; \left[\begin{array}{ccccc} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & -1 & -2 & -3 \end{array}\right] \; R_1 + 2R_2$$

$$\left[\begin{array}{ccccc} 1 & 2 & 0 & -5 & -4 \\ 0 & 0 & -1 & -2 & -3 \end{array}\right] \; (-1)\,R_2 \; \left[\begin{array}{ccccc} 1 & 2 & 0 & -5 & -4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

Thus,

$$\left[\begin{array}{c} x \\ y \\ z \\ u \end{array}\right] = \left[\begin{array}{l} -2s + 5t - 4 \\ s \\ -2t + 3 \\ t \end{array}\right] \text{ where } t \text{ is any real number.}$$

2. Given that $\left(A^T - 2I\right)^{-1} = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array}\right]$. Find A.

Solution:

Taking inverses of both sides we get $A^T - 2I = 2\left[\begin{array}{cc} 3 & -1 \\ -2 & 1 \end{array}\right]$, and so

$$A = \left(2\left[\begin{array}{cc} 3 & -1 \\ -2 & 1 \end{array}\right] + 2\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)^T = \left[\begin{array}{cc} 8 & -4 \\ -2 & 4 \end{array}\right]$$

3. Let $A = GFE$ where $G = \left[\begin{array}{cc} 1 & 0 \\ 0 & -\frac{1}{2} \end{array}\right]$, $E = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$ and $F = \left[\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array}\right]$.
Note that G, E and F are elementary matrices.

(a) Find A^{-1}.

Solution:

$$A^{-1} = (GFE)^{-1} = E^{-1}F^{-1}G^{-1}$$

$$= \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & -2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 2 & -2 \end{array}\right] = \left[\begin{array}{cc} 2 & -2 \\ 1 & 0 \end{array}\right]$$

(b) Solve the system $AX = B$ where $B = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$.

Solution:

$$X = A^{-1}B = \left[\begin{array}{cc} 2 & -2 \\ 1 & 0 \end{array}\right]\left[\begin{array}{c} 1 \\ -1 \end{array}\right] = \left[\begin{array}{c} 4 \\ 1 \end{array}\right]$$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ y^2 \end{array}\right]$ for any real numbers x and y. Show that T is not a linear transformation.

Solution:

Since $T\left(2\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right) = T\left(\left[\begin{array}{c} 0 \\ 2 \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ 4 \end{array}\right]$ and $2T\left(\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right) = 2\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 2 \end{array}\right]$, we see that $T\left(2\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right) \neq 2T\left(\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right)$ and therefore T is not a linear transformation.

5. Let $A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$. Find all values of the number x so that A is not invertible.

Solution:

$$\det A = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \begin{matrix} \\ R_2 - xR_1 \\ R_3 - xR_1 \end{matrix} = \begin{vmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{vmatrix} = (1-x)^2 \begin{vmatrix} 1 & x & x \\ 0 & 1+x & x \\ 0 & x & 1+x \end{vmatrix} =$$

$$(1-x)^2 \begin{vmatrix} 1+x & x \\ x & 1+x \end{vmatrix} = (1-x)^2 (2x+1).$$

Thus, A is not invertible when $detA = 0$, that is, when $x = 1$ or $x = -\frac{1}{2}$.