

# Noether Theorem

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**Definition 1** A diffeomorphism  $\varphi$  of  $M$  is a symmetry of a Lagrangian  $L : TM \rightarrow \mathbb{R}$  if  $(T\varphi)^*L = L$ .

Note that  $T\varphi : TM \rightarrow TM$  is defined as follows. For every  $u \in T_xM$  and  $f \in C^\infty(M)$ ,

$$(T\varphi(u))(f) = u(\varphi^*f) = u(f \circ \varphi).$$

Consider local coordinates  $(q^i)$  on  $M$ . If  $x \in M$  has coordinates  $(q^1(x), \dots, q^n(x))$ , then  $\varphi(x)$  has coordinates

$$(q^1(\varphi(x)), \dots, q^n(\varphi(x))) = (F^1((q^1(x), \dots, q^n(x))), \dots, F^n(q^1(x), \dots, q^n(x))).$$

In other words,  $\varphi$  maps a point with coordinates  $(q^1, \dots, q^n)$  to a point with coordinates  $(F^1((q^1, \dots, q^n)), \dots, F^n(q^1, \dots, q^n))$ . Treating as a point in  $M$  we can write

$$\varphi(q^1, \dots, q^n) = (F^1((q^1, \dots, q^n)), \dots, F^n(q^1, \dots, q^n))$$

Let  $\gamma : [a, b] \rightarrow M : t \mapsto \gamma(t)$  be a curve in  $M$  with coordinates  $(q^1(t), \dots, q^n(t))$ . The tangent vector to the curve  $\gamma$  at  $t$  is

$$\dot{\gamma}(t) = \dot{q}^1 \frac{\partial}{\partial q^1} + \dots + \dot{q}^n \frac{\partial}{\partial q^n}.$$

The vector  $T\varphi(\dot{\gamma}(t))$  is tangent at  $t$  to the curve  $\varphi \circ \gamma : [a, b] \rightarrow M : t \mapsto \varphi(\gamma(t))$ . But

$$(q^1(\varphi(\gamma(t))), \dots, q^n(\varphi(\gamma(t)))) = (F^1(q^1(\gamma(t)), \dots, q^n(\gamma(t))), \dots, F^n(q^1(\gamma(t)), \dots, q^n(\gamma(t)))).$$

Differentiating with respect to  $t$  we get

$$T\varphi(\dot{\gamma}(t)) = \frac{d}{dt}\varphi(\gamma(t)) = \sum_{i,j=1}^n \frac{\partial F^i}{\partial q^j} \frac{dq^j}{dt} \frac{\partial}{\partial q^i} = \sum_{i,j=1}^n \frac{\partial F^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i}.$$

Thus,  $T\varphi$  takes a vector  $\dot{q}^1 \frac{\partial}{\partial q^1} + \dots + \dot{q}^n \frac{\partial}{\partial q^n}$  at  $x \in M$  to the vector  $\sum_{i,j=1}^n \frac{\partial F^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i}$  at  $\varphi(x)$ .

Consider now a 1-parameter group  $\varphi_t = \exp tX$  of diffeomorphisms of  $M$  generated by a vector field

$$X = f^1(q^1, \dots, q^n) \frac{\partial}{\partial q^1} + \dots + f^n(q^1, \dots, q^n) \frac{\partial}{\partial q^n}.$$

Then,

$$F^i(q^1, \dots, q^n) = q^i + t f^i(q^1, \dots, q^n) + \text{terms of order } h \text{ and higher.}$$

Therefore, setting

$$\delta_j^i = \frac{\partial q^i}{\partial q^j}$$

we get

$$\begin{aligned} T\varphi(\dot{q}) &= T(\exp tX)(\dot{q}) = \sum_{i,j=1}^n \frac{\partial F^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i} = \sum_{i,j=1}^n (\delta_j^i + t \frac{\partial f^i}{\partial q^j}) \dot{q}^j \frac{\partial}{\partial q^i} + \text{higher order} \\ &= \sum_{i=1}^n \dot{q}^i \frac{\partial}{\partial q^i} + t \sum_{i,j=1}^n \frac{\partial f^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i} + \text{higher order.} \end{aligned}$$

We are going to compute

$$\begin{aligned} (T \exp(tX))^* L(q, \dot{q}) &= L((\exp(tX)(q), T \exp(tX)(\dot{q})) \\ &= L(q + t f(q), \sum_{i=1}^n \dot{q}^i \frac{\partial}{\partial q^i} + t \sum_{i,j=1}^n \frac{\partial f^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial q^i}) + \text{higher order.} \\ &= L(q + t f(q), \dot{q} + t Df(\dot{q})) + \text{higher order.} \end{aligned}$$

Consider now the action integral  $I = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt$ . Since  $L = L$ , it follows that

$$\begin{aligned} I &= \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt = \int_{t_0}^{t_1} (T(\exp sX))^* L(q(t), \dot{q}(t)) dt \\ &= \int_{t_0}^{t_1} L(q(t) + s f(q(t)), \dot{q}(t) + s(Df)(\dot{q}(t))) dt + \text{higher order in } s. \end{aligned}$$

Differentiating with respect to  $s$  and setting  $s = 0$ , we get

$$0 = \int_{t_0}^{t_1} \left( \frac{\partial}{\partial q} L(q(t), \dot{q}(t)) f(q(t)) + \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) Df(\dot{q}(t)) \right) dt.$$

However,

$$(Df)(\dot{q}(t)) = \frac{d}{dt} f(q(t)).$$

Hence,

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} \left( \frac{\partial}{\partial q} L(q(t), \dot{q}(t)) f(q(t)) + \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) \frac{d}{dt} f(q(t)) \right) dt \\
&= \int_{t_0}^{t_1} \left( \frac{\partial}{\partial q} L(q(t), \dot{q}(t)) - \frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) \right) f(q(t)) dt + \\
&\quad + \frac{\partial}{\partial \dot{q}} L(q(t_1), \dot{q}(t_1)) f(q(t_1)) - \frac{\partial}{\partial \dot{q}} L(q(t_0), \dot{q}(t_0)) f(q(t_0)).
\end{aligned}$$

The integrand vanishes if the curve  $t \mapsto q(t)$  satisfies the Lagrange equations. In this case

$$\frac{\partial}{\partial \dot{q}} L(q(t_1), \dot{q}(t_1)) f(q(t_1)) = \frac{\partial}{\partial \dot{q}} L(q(t_0), \dot{q}(t_0)) f(q(t_0)).$$

Therefore,  $\frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) f(q(t))$  is constant along any curve  $t \mapsto q(t)$  satisfying the Lagrange equations. Thus, is a constant of motion. It is denoted by  $J_X(q, \dot{q})$ .

It remains to write the obtained constant of motion in geometric terms. Introducing the Legendre transformation  $\mathfrak{L} : TM \rightarrow T^*M : (q, \dot{q}) \mapsto (q, p)$  where  $p = \frac{\partial}{\partial \dot{q}} L(q, \dot{q})$ , we can pull-back the canonical 1-form  $\theta = pdq$  of  $T^*M$  to  $TM$  obtaining

$$\mathfrak{L}^* \theta = \frac{\partial}{\partial \dot{q}} L(q, \dot{q}) dq.$$

Clearly,

$$J_X = \mathfrak{L}^* \theta(X) = \langle \mathfrak{L}^* \theta \mid X \rangle.$$