

Pmat 421
Assignment # 1

$$\begin{aligned}
 1. \left(\frac{2+i}{3+4i} + \frac{2i}{3-4i} \right)^2 &= \left(\frac{(2+i)(3-4i) + 2i(3+4i)}{(3+4i)(3-4i)} \right)^2 = \\
 &= \left(\frac{6-8i+3i+4+6i-8}{3^2+4^2} \right)^2 = \\
 &= \left(\frac{2+i}{25} \right)^2 = \frac{1}{625} (2+4i-1) = \frac{3}{625} + i \frac{4}{625}.
 \end{aligned}$$

2. For $z = x + iy, w = a + ib, \quad z\bar{w} = (x + iy)(a - ib) = (ax + by) + i(ay - bx)$

and $\bar{z}w = (x - iy)(a + ib) = (ax + by) + i(-ay + bx)$

they are equal if real parts are the same and imaginary parts as well

$$(ay - bx) = - (ay - bx) \text{ thus } (ay - bx) = 0$$

it means both real or if $by \neq 0$

$$\text{or } \frac{a}{b} = \frac{x}{y} = r \quad z = y(r + i) \quad w = b(r + i)$$

i.e. $w = sz$, where s is real, $s = \frac{b}{y}$.

3. $|z + i| \leq |z + 2| \rightarrow x^2 + (y + 1)^2 \leq (x + 2)^2 + y^2$

$$2y + 1 \leq 4x + 4 \quad y \leq 2x + \frac{3}{2} \quad \text{the set is below and on the line,}$$

the boundary is the line included, so the set is closed and simply connected

4. $\{(\operatorname{Re} z)^2 > 1\} = \{|x| > 1\} = \{x > 1\} \cup \{x < -1\}$

the boundary is two vertical lines $x = \pm 1$, accm.pts $S' = \{(\operatorname{Re} z)^2 \geq 1\}$

the set is open and not connected

5. $-1 + i = \sqrt{2}e^{i\frac{3}{4}\pi} \quad (-1 + i)^8 = (\sqrt{2})^8 e^{i\frac{3}{4}\pi \cdot 8} = 2^4 e^{i6\pi} = 2^4 = 16$

$$\begin{aligned}
 1 - i\sqrt{3} &= 2 \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) = 2e^{-i\frac{\pi}{3}} \quad (1 - i\sqrt{3})^5 = 2^5 e^{-i\frac{\pi}{3} \cdot 5} = 2^5 e^{i\frac{\pi}{3}(1-6)} = \\
 &= 2^5 e^{i\frac{\pi}{3}} e^{-i2\pi} = 2^5 e^{i\frac{\pi}{3}}
 \end{aligned}$$

$$:(-1 + i)^8 (1 - i\sqrt{3})^5 = 2^4 2^5 e^{i\frac{\pi}{3}} = 2^9 (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = 2^8 + 2^8 \sqrt{3}i$$

$$\text{check: } (-1 + i)^8 = \left((-1 + i)^2 \right)^4 = (2i)^4 = 2^4 (-1)^2 = 2^4$$

$$\begin{aligned}
 (1 - i\sqrt{3})^5 &= (1 - i\sqrt{3}) \left[(1 - i\sqrt{3})^2 \right]^2 = (1 - i\sqrt{3}) [(-2 - 2i\sqrt{3})]^2 = \\
 &= (1 - i\sqrt{3}) 4 (1 + i\sqrt{3})^2 = 4 (1 - i\sqrt{3}) (-2 + 2i\sqrt{3}) = -8 (1 - i\sqrt{3})^2 = \\
 &= 2^4 (1 + i\sqrt{3})
 \end{aligned}$$

6. $\left(\frac{1+i}{1-i}\right)^3 - 2i = \left(\frac{(1+i)^2}{1+1}\right)^3 - 2i = i^3 - 2i = -3i$ so $Arg = -\frac{\pi}{2}$, generally

$$\left(\frac{1+i}{1-i}\right)^3 - 2i = 3e^{i\pi(\frac{1}{2}+2k)}, k \text{ is any integer,}$$

then $\sqrt{\left(\frac{1+i}{1-i}\right)^3 - 2i} = \sqrt{3}e^{i\pi(-\frac{1}{4}+k)} = \pm\sqrt{3}e^{-i\pi\frac{1}{4}} = \pm\sqrt{\frac{3}{2}}(1-i)$

7. For $z = |z|e^{i\theta}$ where $\theta = Arg(z)$ must be in $(-\pi, \pi]$ $Arg(\bar{z}) = -Arg(z)$.
 $\bar{z} = |z|e^{-i\theta}$ so $-\theta$ is not Arg only if $\theta = \pi$, the formula is valid for all z except $z = x \leq 0$;

similarly if both $Arg(z), Arg(w) \in (\frac{\pi}{2}, \pi)$ then $Arg(z) + Arg(w) \in (\pi, 2\pi)$ not Arg

generally it must $Arg(z) + Arg(w) = Arg(zw) + 2k\pi$ for some value of k .
 similarly for $(-\pi, -\frac{\pi}{2})$.

8. we have to prove two implications: if $\arg z = \arg w$ then $|z+w| = |z|+|w|$
 and if $|z+w| = |z|+|w|$ then $\arg z = \arg w$ which is equivalent to
 if $\arg z \neq \arg w$ then $|z+w| \neq |z|+|w|$

Case I: if $\arg z = \arg w$ then $z = rw$ for some $r > 0$ and $|z+w| = (r+1)|w| = |z|+|w|$

Case II: if $\arg z \neq \arg w$ and $z = -rw, r > 0$ then $|z+w| = |1-r||w| \neq (1+r)|w|$

Case III: if $\arg z \neq \arg w$ and $z \neq \pm rw$ then we can draw a triangle with sides being vectors z and w ,

and the third side being $z+w$ (vector addition), clearly $|z+w| \neq |z|+|w|$

OR

$|z+w| = |z|+|w|$ iff $|z+w|^2 = (|z|+|w|)^2$ iff $\operatorname{Re}(z\bar{w}) = |z\bar{w}|$ since

$$\begin{aligned} \text{L.S. } (z+w)(\bar{z}+\bar{w}) &= |z|^2 + z\bar{w} + \bar{z}w + |w|^2 = |z|^2 + z\bar{w} + \bar{z}w + |w|^2 = \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \end{aligned}$$

$$\text{R.S. } (|z|+|w|)^2 = |z|^2 + 2|z||w| + |w|^2 = |z|^2 + 2|z\bar{w}| + |w|^2$$

and $\operatorname{Re}(z\bar{w}) = |z\bar{w}|$ iff $\operatorname{Re}(z\bar{w}) > 0$ and $\operatorname{Im}(z\bar{w}) = 0$ iff $\arg(z\bar{w}) = 0$

and $\arg(z\bar{w}) = \arg(z) + \arg(\bar{w}) = \arg(z) - \arg(w) = 0$

9. Solving $z^5 = -1$; express $-1 = e^{i\pi(1+2k)}$ then $(-1)^{\frac{1}{5}} = e^{i\frac{\pi}{5}(1+2k)}$
 we need 5 different values of k

$$k = 0 \quad z_1 = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$$

$$k = 2 \quad z_2 = e^{i\pi} = -1$$

$$k = 1 \quad z_3 = \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}$$

$$k = -1 \quad z_4 = e^{-i\frac{\pi}{5}} = \bar{z}_1 = \cos \frac{\pi}{5} - i \sin \frac{\pi}{5}$$

$$k = -2 \quad z_5 = e^{-3i\frac{\pi}{5}} = \bar{z}_3 = \cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5}$$

the angle between roots is 72 degrees or $\frac{2}{5}\pi$ and we have a symmetry -reflection in the x -axis

10. De Moivre's Theorem: $R.S. \quad e^{i\theta 5} = \cos(5\theta) + i \sin(5\theta)$

$$L.S. \quad (e^{i\theta})^5 = (\cos \theta + i \sin \theta)^5 = \sum_{k=0}^{k=5} \binom{5}{k} \cos^{5-k} \theta \cdot i^k \sin^k \theta$$

we need to compare the imaginary parts i.e. only odd $k = 1, 3, 5$.

$$i^3 = -i \quad i^5 = i \quad \binom{5}{3} = 10, \binom{5}{1} = 5, \binom{5}{5} = 1$$

$$\sin(5\theta) = 5 \cos^4 \theta \cdot \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$