

PMAT 421 WINTER 00
FINAL
SOLUTIONS

For 1a)

$$(-1)^{1-i} = e^{(1-i)\log(-1)} = e^{(1-i)[\ln 1 + i(\pi + 2k\pi)]} = e^{i(\pi + 2k\pi)} \cdot e^{\pi(2k+1)} = -e^{\pi(2k+1)} \quad \text{for any integer } k.$$

For 1b)

$$\sin(i - \pi) = \frac{1}{2i} (e^{i(i-\pi)} - e^{-i(i-\pi)}) = \frac{-i}{2} (e^{-1}e^{-i\pi} - e^1e^{i\pi}) = \frac{i}{2} \left(\frac{1}{e} - e \right).$$

For 2)

for $\sin z = -i$ we can use the definition of $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i$

then $e^{iz} - e^{-iz} = 2$; for $w = e^{iz}$ we get $w - \frac{1}{w} = 2$

and $w^2 - 2w - 1 = 0$

$w_{1,2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$ or $w_1 = \sqrt{2} + 1 > 0, w_2 = 1 - \sqrt{2} < 0$

$z_k = \frac{1}{i} \log w_1 = -i [\ln(\sqrt{2} + 1) + i(2k\pi)] = 2k\pi - i \ln(\sqrt{2} + 1)$

AND

$z_l = \frac{1}{i} \log w_2 = -i [\ln(\sqrt{2} - 1) + i(\pi + 2l\pi)]$

$= -i \ln(\sqrt{2} - 1) + (2l + 1)\pi = (2l + 1)\pi + i \ln(\sqrt{2} + 1)$

for any integers k, l .

For 3)

the property of \log :

$\log z = -\log \frac{1}{z} + i2n\pi$ for certain $n, z \neq 0$

a)

to get $n = 0$ we need both $\text{Arg } z, \text{Arg } \frac{1}{z} \in (-\pi, \pi]$ but

$\text{Arg } \frac{1}{z} = -\arg z \in [-\pi, \pi)$

so we have to **exclude points on the cut** $z = x \leq 0$

where $\text{Arg } z = \pi$

b)

it means that $\arg z \in [0, 2\pi)$ then $\arg \frac{1}{z} \in (-2\pi, 0]$

so **only** for points on the cut where $\arg z = \arg \frac{1}{z} = -\arg z = 0$

i.e. $z = x > 0$ the relation is valid.

For 4)

$f(z) = \frac{z}{z+4}$ is analytic for $z \neq -4$ distance from i to -4

is $|i+4| = \sqrt{17}$ thus around $z_0 = i$ we have two possible domains

$|z-i| < \sqrt{17}$ or $|z-i| > \sqrt{17}$, distance from i to 10 is $\sqrt{101}$

thus we need the latter to get 10 in, so negative powers of $(z-i)$:

first

$$\frac{1}{z+4} = \frac{1}{z-i+4+i} = \frac{1}{z-i} \cdot \frac{1}{1 + \frac{4+i}{z-i}} = \left(\text{for } \left| \frac{4+i}{z-i} \right| < 1 \right)$$

$$= \frac{1}{z-i} \sum_{n=0}^{\infty} \frac{(-1)^n (4+i)^n}{(z-i)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (4+i)^n}{(z-i)^{n+1}}$$

now

$$\frac{z}{z+4} = \frac{z+4-4}{z+4} = 1 - \frac{4}{z+4} =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{4(-1)^n (4+i)^n}{(z-i)^{n-1}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4(4+i)^{k-1}}{(z-i)^k}$$

for $|z-i| > \sqrt{17}$ and $b_2 = 4(4+i)$

For 5)

Is $|\sin z| \leq 1$ for all complex z ? The function $\sin z$ is entire i.e. analytic everywhere if it is bounded then by Liouville's Th. $\sin z = \text{const} \dots$ contradiction

"If f is entire and bounded i.e. analytic and for some $M > 0$ $|f(z)| \leq M$ for **all** z ; then f is constant."

Also $|\sin in| = \frac{1}{2} |e^{-n} - e^n| \rightarrow \infty$ as $n \rightarrow \infty$

For 6)

$\int_c \frac{1}{\sqrt{z}} dz$ where c is the curve from $-i$ to $1+i$ lying in the right half of the plane

we can parametrize a line segment or use Fundamental Th.

$$F(z) = 2\sqrt{z} = 2e^{\frac{1}{2}\text{Log}z}$$

in the complex plane excluding the cut =negative x-axis

$$\text{since } F'(z) = \frac{1}{\sqrt{z}}$$

$$\text{so } \int_c \frac{1}{\sqrt{z}} dz = F(1+i) - F(-i) = 2 \left[(1+i)^{\frac{1}{2}} - (-i)^{\frac{1}{2}} \right]$$

principal branch means that $\text{Arg}(1+i) = \frac{\pi}{4}$ and $\text{Arg}(-i) = -\frac{\pi}{2}$ so

$$(1+i)^{\frac{1}{2}} = e^{\frac{1}{2}\text{Log}(1+i)} = e^{\frac{1}{2} \ln \sqrt{2} + i \frac{\pi}{8}} = \sqrt[4]{2} \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right)$$

$$\text{and } (-i)^{\frac{1}{2}} = e^{\frac{1}{2}\text{Log}(-i)} = e^{-i \frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\text{finally } \int_c \frac{1}{\sqrt{z}} dz = 2^{\frac{5}{4}} \cos \frac{\pi}{8} - \sqrt{2} + i(2^{\frac{5}{4}} \sin \frac{\pi}{8} + \sqrt{2}).$$

For 7)

$f(z) = \frac{1}{z} e^{z+\frac{2}{z}}$ has an essential singularity at $z=0$

$$f(z) = \frac{1}{z} e^z e^{\frac{2}{z}} = \frac{1}{z} \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots \right) \left(1 + \frac{2}{z} + \frac{2}{z^2} + \frac{4}{3z^3} \right)$$

$$\text{so the residue: } 1 + 2 + \frac{2^2}{(2!)^2} + \frac{2^3}{(3!)^2} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{(n!)^2}.$$

For 8)

$$\int_0^{\infty} \frac{\cos \frac{\pi}{4} x}{x^4 - 16} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos \frac{\pi}{4} x}{x^4 - 16} dx = \frac{1}{2} \text{Re} \int_{-\infty}^{\infty} \frac{e^{ix \frac{\pi}{4}}}{x^4 - 16} dx$$

$$f(z) = \frac{1}{z^4 - 16} = \frac{1}{(z^2 - 4)(z^2 + 4)}$$
 is analytic

except at $z = \pm 2 \dots$ on the real axis and $z = \pm 2i$, only $z = +2i$ inside the curve

the curve $c = [-R, -2 - \epsilon] \cup c_{\epsilon}^{-2} \cup [-2 + \epsilon, 2 - \epsilon] \cup c_{\epsilon}^2 \cup [2 + \epsilon, R] \cup \Gamma_R$

with two "dents" - semicircles around $z = \pm 2, R \gg 2$

$$\oint_c f(z) e^{iz \frac{\pi}{4}} dz = 2\pi i \text{Res}_{2i}, \text{ since } \lim_{z \rightarrow \infty} f(z) = 0 \text{ as } R \rightarrow \infty,$$

$$\text{and we know the limits as } \epsilon \rightarrow 0^+ \text{ of } \oint_{c_{\epsilon}} f(z) e^{iz \frac{\pi}{4}} dz \rightarrow -\pi i [\text{Res}_{-2} + \text{Res}_2]$$

$$\int_0^{\infty} \frac{\cos \frac{\pi}{4}x}{x^4 - 16} dx = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix\frac{\pi}{4}}}{x^4 - 16} dx = \frac{1}{2} \operatorname{Re}\{2\pi i \operatorname{Res}_{2i} + \pi i [\operatorname{Res}_{-2} + \operatorname{Res}_2]\} =$$

$$= -\frac{\pi}{32}(e^{-\frac{\pi}{2}} + 1)$$

$$\operatorname{Res}_{2i} = \left[(z - 2i) f(z) e^{iz\frac{\pi}{4}} \right]_{z=2i} = \left[\frac{e^{iz\frac{\pi}{4}}}{(z^2 - 4)(z + 2i)} \right]_{z=2i} = \frac{e^{-\frac{\pi}{2}}}{-32i}$$

$$\operatorname{Res}_2 = \left[(z - 2) f(z) e^{iz\frac{\pi}{4}} \right]_{z=2} = \left[\frac{e^{iz\frac{\pi}{4}}}{(z^2 + 4)(z + 2)} \right]_{z=2} = \frac{e^{i\frac{\pi}{2}}}{32} = \frac{i}{32} = \frac{-1}{i32}$$

$$\operatorname{Res}_{-2} = \left[(z + 2) f(z) e^{iz\frac{\pi}{4}} \right]_{z=-2} = \left[\frac{e^{iz\frac{\pi}{4}}}{(z^2 + 4)(z - 2)} \right]_{z=-2} = \frac{e^{-i\frac{\pi}{2}}}{-32} = \frac{-i}{-32} = \frac{-1}{i32}$$

For 9)

$$\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3 \sin \theta} d\theta = \operatorname{Im} \int_0^{2\pi} \frac{e^{i3\theta}}{5 - 3 \sin \theta} d\theta = I$$

by subst. $z = e^{i\theta}$, $\frac{dz}{iz} = d\theta$, $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$, $3^{i3\theta} = z^3$

$$\oint_{|z|=1} \frac{z^3}{5 - 3 \cdot \frac{z^2 - 1}{2iz}} \frac{dz}{iz} = \oint_{|z|=1} \frac{z^3}{5iz - \frac{3}{2} \cdot (z^2 - 1)} dz = \oint_{|z|=1} \frac{2z^3 dz}{10iz - 3z^2 + 3} =$$

$$= \oint_{|z|=1} \frac{-2z^3 dz}{3z^2 - 10iz - 3} \quad \text{find the poles : } 3z^2 - 10iz - 3 = 0$$

$$z_{1,2} = \frac{10i \pm \sqrt{-100 + 36}}{6} = \frac{10i \pm 8i}{6} = \frac{1}{3}i \text{ or } 3i \text{ and } 3z^2 - 10iz - 3 = 3 \left(z - \frac{i}{3} \right) (z - 3i)$$

only $z_1 = \frac{i}{3}$ is inside the unit circle so by Residue Th.

$$\oint_{|z|=1} \frac{-2z^3 dz}{3z^2 - 10iz - 3} = 2\pi i \operatorname{Res}_{\frac{i}{3}} = 2i\pi \left[\left(z - \frac{i}{3} \right) f(z) \right]_{z=\frac{i}{3}} = 2\pi i \left[\frac{-2z^3}{3(z - 3i)} \right]_{z=\frac{i}{3}} =$$

$$= 2\pi i \cdot \frac{-2 \left(\frac{i}{3} \right)^3}{-8i} = \frac{\pi}{2} \cdot \frac{-i}{27} \text{ and } I = \operatorname{Im} \oint_{|z|=1} \dots = \frac{-\pi}{54}.$$

For 10a)

the mapping $w = f(z) = z - \frac{1}{z}$ is conformal if $z \neq \pm i, 0$

since $f'(z) = 1 + \frac{1}{z^2} = \frac{z^2 + 1}{z^2} \neq 0$

$w = f(\pm i) = \pm 2i$ at the critical points

For b)

for $|z| = 2$ $z(t) = 2e^{it}$ so $w = 2e^{it} - \frac{1}{2}e^{-it} = \frac{3}{2} \cos \theta + i\frac{5}{2} \sin \theta$

$\left(\frac{2}{3}u \right)^2 + \left(\frac{2}{5}v \right)^2 = 1$ an ellipse with $\pm 2i$ inside

For c)

if $z = iy$, $y \neq 0$ then $w = iy - \frac{1}{iy} = i\frac{y^2+1}{y}$ $u = 0$, $v \neq 0$

so y-axis minus origin onto a part of v-axis minus origin

to find out which part find the range of $f(t) = \frac{t^2 + 1}{t}$

$$\text{or } \left| \frac{t^2 + 1}{t} \right| \geq 2 \quad t^2 + 1 - 2|t| = (|t| - 1)^2 \geq 0$$

thus $|v| \geq 2$ $z = \pm i \rightarrow w = \pm 2i$

For d)

$|z| = 1$ passes through $\pm i$ and $z = e^{i\theta}$

$w = e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ so $u = 0, v \in [-2, 2]$.. vertical line segment

Notice that y-axis and unit circles go through the critical points and are perpendicular to each other at $z = \pm i$ but images are parallel.