

PMAT 421 WINTER 99
FINAL 3 hours

1. Find all values of $\left(-\frac{\pi}{2}\right)^{\frac{1}{i}}$ in the form $a + ib$ where a, b are real numbers. [5]
2. Find all solutions of $\sin z = 2$ in the form $a + ib$ where a, b are real numbers. [9]
3. Explain Bernoulli's paradox:
Since $(-z)^2 = z^2$ then for $z \neq 0$ $\text{Log}(-z)^2 = \text{Log } z^2$ and
 $2\text{Log}(-z) = 2\text{Log } z$ and by cancelling $2\text{Log}(-z) = \text{Log } z$.
Find the error in the reasoning! [5]
4. Find the Laurent series of $f(z) = \frac{1}{(z-4)^2}$ around $z_0 = 1$
in the domain containing the point -4 . [9]
5. Explain why for any $R > 0$ there are infinitely many z such that
 $|z| > R$ and $|\cos z| > 100$.
State the theorem used. [7]
6. Evaluate $\int_c \text{Im } z \, dz$ where c is a part of the circle centered at i with radius 1
between the origin and $1 + i$. [9]
7. For $f(z) = \frac{1}{z \sin z}$
 - (a) classify all singularities;
 - (b) find the residue at $z_0 = 0$. [9]
8. Evaluate $\int_0^{\infty} \frac{x^2}{x^4 + 4} dx$ by means of Residue Theorem. Explain all your steps. [9]
9. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta$ by means of Residue Theorem. Explain all your steps. [9]
10. For $w = \log z$ with $\frac{\pi}{4} \leq \arg z < \frac{9\pi}{4}$ find
 - (a) where the mapping is conformal;
 - (b) the image of the y axis without the origin;
 - (c) the image of the unit circle in the w plane. [9]

SOLUTIONS

For 1)

$$\begin{aligned} \left(-\frac{\pi}{2}\right)^{\frac{1}{i}} &= e^{\frac{1}{i} \log\left(\frac{-\pi}{2}\right)} = e^{\frac{1}{i} [\ln \frac{\pi}{2} + i(\pi + 2k\pi)]} = e^{-i \ln \frac{\pi}{2}} \cdot e^{\pi(2k+1)} = \\ &= e^{\pi(2k+1)} \cos\left(\ln \frac{\pi}{2}\right) - i e^{\pi(2k+1)} \sin\left(\ln \frac{\pi}{2}\right) \text{ for any integer } k. \end{aligned}$$

For 2)

we can use the definition of \sin : $\frac{e^{iz} - e^{-iz}}{2i} = 2$ and $w = e^{iz}$

$$\text{we get } w - \frac{1}{w} = 4i \text{ and then } w^2 - 4iw - 1 = 0$$

$$w_{1/2} = \frac{4i \pm \sqrt{-16+4}}{2} = i(2 \pm \sqrt{3}) \text{ or } w_1 = i(2 + \sqrt{3}), w_2 = \frac{i}{2 + \sqrt{3}}$$

now

$$z = \frac{1}{i} \log w = -i \left[\ln(2 \pm \sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right) \right] = \frac{\pi}{2} + 2k\pi \pm i \ln(2 + \sqrt{3})$$

for any integer k .

For 3)

the property of \log : $\log w^2 = 2 \log w + i2n\pi$ for certain n ONLY

it means that $2 \log(-z) = 2 \log z + i2n\pi$

in detail if $|\text{Arg } z| > \frac{\pi}{2}$ i.e. the point z is in the left half

$2 \cdot |\text{Arg } z| > \pi$ so it is not principal value anymore thus in this case

$$\text{Arg } z^2 \neq 2 \cdot \text{Arg } z.$$

For 4)

the center is 1 and singular point is 4 so we have two possible domains

$|z - 1| < 3$ or $|z - 1| > 3$ we need the latter so negative powers of $(z - 1)$:

first

$$\begin{aligned} \frac{1}{z-4} &= \frac{1}{(z-1)-3} = \frac{1}{z-1} \cdot \frac{1}{1-\frac{3}{z-1}} = \left(\text{for } \left| \frac{3}{z-1} \right| < 1 \right) \\ &= \frac{1}{z-1} \sum_{n=0}^{\infty} \frac{3^n}{(z-1)^n} = \sum_{n=0}^{\infty} \frac{3^n}{(z-1)^{n+1}} = \sum_{k=1}^{\infty} 3^{k-1} (z-1)^{-k} \end{aligned}$$

differentiate

$$\frac{-1}{(z-4)^2} = \sum_{k=1}^{\infty} 3^{k-1} (-k) (z-1)^{-k-1}$$

and for $|z - 1| > 3$

$$\frac{1}{(z-4)^2} = \sum_{k=1}^{\infty} 3^{k-1} (k) (z-1)^{-k-1} = \sum_{k=1}^{\infty} \frac{k 3^{k-1}}{(z-1)^{k+1}} = \sum_{n=2}^{\infty} \frac{(n-1) 3^{n-2}}{(z-1)^n}.$$

For 5)

the function \cos is 2π -periodic so if we can find one z we have infinitely many.

By contradiction:

if there is no such z it means there is $R_0 > 0$ that $|\cos z| \leq 100$ for all $|z| > R_0$

but \cos is continuous for $|z| \leq R_0$ so $|\cos z| \leq M$ for some $M > 0$

together

\cos is entire and bounded so by Liouville's Th. $\cos z = \text{const.}$ which is NOT true.

Liouville's Th.:

If f is entire and bounded i.e $f'(z)$ exists for all z and for some $M > 0$ $|f(z)| \leq M$ for all z ;

then f is constant .

For 6)

since $\text{Im } z$ is nowhere analytic we have to use the definition of the integral

first parametrize the curve $c : |z - i| = 1$

$$z(t) = i + e^{it}, t \in \left[-\frac{\pi}{2}, 0\right], \text{Im } z(t) = 1 + \sin t, dz = ie^{it} dt$$

$$\int_c \text{Im } z \, dz = \int_{-\frac{\pi}{2}}^0 (1 + \sin t) i (\cos t + i \sin t) dt =$$

$$= i \int_{-\frac{\pi}{2}}^0 [\cos t + \cos t \sin t] dt - \int_{-\frac{\pi}{2}}^0 (\sin t + \sin^2 t) dt =$$

$$= \left[i \sin t + i \frac{\sin^2 t}{2} + \cos t \right]_{-\frac{\pi}{2}}^0 - \int_{-\frac{\pi}{2}}^0 \frac{1 - \cos 2t}{2} dt = 1 + i - \frac{i}{2} - \frac{1}{2} \cdot \frac{\pi}{2} + \left[\frac{\sin 2t}{4} \right]_{-\frac{\pi}{2}}^0 =$$

$$= \left(1 - \frac{\pi}{4}\right) + \frac{1}{2}i.$$

For 7a)

$z \sin z = 0$ for $z = k\pi$ for any integer k

for $k = 0$ $z_0 = 0$ is a pole of order $m = 2$ since $\lim_{z \rightarrow 0} z^2 f(z) = \lim_{z \rightarrow 0} \frac{z}{\sin z} = 1 \neq 0$

for $k \neq 0$ $z_k = k\pi$ are simple poles

$$\text{since } \lim_{z \rightarrow k\pi} (z - k\pi) f(z) = \frac{1}{k\pi} \cdot \lim_{z \rightarrow 0} \frac{z - k\pi}{\sin z} = \frac{1}{k\pi \cos k\pi} = \frac{(-1)^k}{k\pi} \neq 0$$

for b)

$$\text{Res}[f, 0] = \lim_{z \rightarrow 0} [z^2 f(z)]' = \lim_{z \rightarrow 0} \left[\frac{z}{\sin z} \right]' = \lim_{z \rightarrow 0} \frac{\sin z - z \cos z}{\sin^2 z} (L'H) =$$

$$= \lim_{z \rightarrow 0} \frac{\cos z - \cos z + z \sin z}{2 \sin z \cos z} = 0.$$

For 8)

$$\int_0^{\infty} \frac{x^2}{x^4 + 4} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 4} dx = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2}{x^4 + 4} dx$$

we will use Residue Th. for

$$\int_c \frac{z^2}{z^4 + 4} dz = \int_{c_1} \frac{z^2}{z^4 + 4} dz + \int_{c_2} \frac{z^2}{z^4 + 4} dz = 2\pi i \sum \text{Res at all poles inside } c$$

where $c_1 : [-R, R]$ line segment and $c_2 : |z| = R, \text{Im } z > 0$ half circle, for $R > 2$

we can estimate

$$\left| \int_{c_2} \frac{z^2}{z^4 + 4} dz \right| \leq \max_{c_2} \left| \frac{z^2}{z^4 + 4} \right| \cdot \text{length } c_2 \leq \max_{|z|=R} \frac{|z|^2}{|z|^4 - 4} \cdot \pi R \leq \frac{\pi R^3}{R^4 - 4} \rightarrow 0$$

as $R \rightarrow \infty$

Next, find all singularities of the integrand function i.e. solve $z^4 = -4$

$$z^2 = \pm 2i \text{ and } z = \pm(1+i) \text{ or } \pm(1-i) \text{ OR } z = 4^{\frac{1}{4}} e^{i\frac{\pi}{4} + i\frac{\pi k}{2}} = \sqrt{2} e^{i\pi(\frac{1+2k}{4})}$$

inside the curve only poles with $\text{Im } z > 0$ so $z_1 = 1+i, z_2 = -1+i$

$$\text{Res}[f, z_j] = \frac{p(z_j)}{q'(z_j)} = \frac{z_j^2}{4z_j^3} \quad \text{OR}$$

$$\text{Res}[f, z_j] = \lim_{z \rightarrow z_j} (z - z_j) f(z) = \lim_{z \rightarrow z_j} (z - z_j) \frac{z^2}{z^4 + 4} = z_j^2 \cdot \frac{1}{4z_j^3} = \frac{1}{4z_j}$$

$$\text{Re } s[f, z_1] + \text{Re } s[f, z_2] = \frac{1}{4} \left[\frac{1}{1+i} + \frac{1}{-1+i} \right] = -\frac{i}{4}$$

$$\text{and the integral is } 2\pi i \cdot \frac{-i}{4} = \frac{\pi}{2} \text{ and } \int_0^{\infty} \frac{x^2}{x^4 + 4} dx = \frac{\pi}{4}.$$

For 9)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta = \text{Re} \int_0^{2\pi} \frac{e^{i2\theta}}{5 + 4 \cos \theta} d\theta = \text{Re} \oint_{|z|=1} \frac{z^2}{5 + 4 \left(\frac{z^2+1}{2z} \right)} \frac{dz}{iz} =$$

(by subst. $z = e^{i\theta}, \cos \theta = \frac{1}{2}(z + \frac{1}{z})$ and $dz = ie^{i\theta} d\theta, e^{i2\theta} = z^2$)

$$= \text{Re} \frac{1}{i} \oint_{|z|=1} \frac{z^2}{5z + 2z^2 + 2} dz = \text{Re} \frac{1}{i} \oint_{|z|=1} \frac{z^2}{2 \left(z + \frac{1}{2} \right) (z + 2)} dz =$$

$$= \text{Re} \left[\frac{1}{i} \cdot 2\pi i \cdot \text{Res} \left[f, -\frac{1}{2} \right] \right] = \frac{\pi}{6}$$

since only $z = -\frac{1}{2}$ is inside the unit circle and

$$\text{Re } s \left[f, -\frac{1}{2} \right] = \left[\frac{z^2}{2(z+2)} \right]_{z=-\frac{1}{2}} = \frac{1}{12}.$$

For 10a)

the given branch of log is cont. and therefore analytic in the z -plane except

the branch cut : $S = \left\{ \theta = \frac{\pi}{4} \text{ i.e. } \operatorname{Im} z = \operatorname{Re} z \geq 0 \right\} = \{y = x, x \geq 0\}$

so mapping is conformal on $C - S$ since $(\log z)' = \frac{1}{z} \neq 0$ there;

For b)

for $x = 0, y > 0$

$w = \ln y + i\frac{\pi}{2}$horizontal line since $u = \ln y$ is any real #, $v = \frac{\pi}{2}$

for $x = 0, y < 0$

$w = \ln(-y) + i\left(-\frac{\pi}{2} + 2\pi\right)$ horizontal line $v = \frac{3\pi}{2}$

For c)

$|z| = 1$ means $z = e^{i\theta}$ where $\frac{\pi}{4} \leq \theta < \frac{9\pi}{4}$

$w = i\theta$... vert. line segment, $u = 0, \frac{\pi}{4} \leq v < \frac{9\pi}{4}$