

PMAT 421 WINTER 08
Assignment #4 Due by Friday April 11, 4pm:

1. since $z = -1$ is a pole of order 10 of $f(z) = \frac{z^{12}}{(z+1)^{10}}$

$$\begin{aligned} \operatorname{Res}_{-1} f(z) &= \frac{1}{9!} \left[(z+1)^{10} f(z) \right]^{(9)} = \frac{1}{9!} \left[z^{12} \right]^{(9)} \text{ (at } z = -1) = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{9!} z^4 = \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{9!3!} (-1)^4 = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} = 220. \end{aligned}$$

2. The integrand function is analytic except the branch cut $\{\operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$

and where $\sin(z-1) = 0 \quad z-1 = k\pi \quad z = 1 + k\pi$ for an integer k

only two of those are inside the given curve $z_0 = 1$ and $z_1 = 1 + \pi = 4.14\dots$

e.g. $z_2 = 1 + 2\pi > 7 \quad z_{-1} = 1 - \pi$ negative...

we need Residues :but $z = 1$ is a removable singularity since

$$\lim_{z \rightarrow 1} \frac{\operatorname{Log} z}{\sin(z-1)} = \lim_{z \rightarrow 1} \frac{\frac{1}{z}}{\cos(z-1)} = 1 \text{ (L.H.rule) thus Residue is 0;}$$

$z = 1 + \pi$ is a simple pole thus

$$\operatorname{Res}_{1+\pi} f(z) = \lim_{z \rightarrow 1+\pi} \frac{(z-1-\pi) \operatorname{Log} z}{\sin(z-1)} =$$

$$= \operatorname{Log}(1+\pi) \cdot (\text{L.H.rule}) \lim_{z \rightarrow 1+\pi} \frac{1}{\cos(z-1)} = -\ln(1+\pi) \text{ and}$$

$$\int_c \frac{\operatorname{Log} z}{\sin(z-1)} dz = 2\pi i \operatorname{Res}_{1+\pi} f(z) = -2\pi \ln(1+\pi)i.$$

3. since $z = 0$ is an essential singularity of $f(z) = z^3 \cos \frac{2}{z}$ use Laurent series

$$\cos w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} \quad f(z) = z^3 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left(\frac{1}{z}\right)^{2n} = z^3 \left(1 - \frac{2}{z^2} + \frac{2^4}{4!} \cdot \frac{1}{z^4} - \dots \right)$$

$$\text{thus } \operatorname{Res}_0 f(z) = \frac{2^4}{4!} = \frac{2}{3}.$$

4. For all singular points solve $e^{2z} + e^z(1-i) - i = 0$

$$\text{first } w = e^z \quad w^2 + w(1-i) - i = (w-i)(w+1) = 0$$

$$\text{so } w_1 = i \text{ and } w_2 = -1 \quad e^z = i \text{ and } e^z = -1$$

$$\text{two groups } z_k = \log i = \frac{\pi}{2}i + 2k\pi i \quad z_n = \log(-1) = i\pi(1+2n)$$

where k and n are any integers; all are simple poles

$$\operatorname{Res}_{z_k} f(z) = \left[\frac{1}{(e^{2z} + e^z(1-i) - i)'} \right]_{z_k} =$$

$$\left[\frac{1}{2e^{2z} + e^z(1-i)} \right]_{z_k} = \frac{1}{2(i)^2 + i(1-i)} = \frac{1}{-1+i} = \frac{-1-i}{2}$$

$$\operatorname{Re} s_{z_n} f(z) = \left[\frac{1}{(e^{2z} + e^z(1-i) - i)'} \right]_{z_n} =$$

$$\left[\frac{1}{2e^{2z} + e^z(1-i)} \right]_{z_n} = \frac{1}{2(-1)^2 - 1(1-i)} = \frac{1}{1+i} = \frac{1-i}{2}.$$

5. First all singular points of the integrand function $e^z = -e^{-z}$ $e^{2z} = -1$

$$\text{so } 2z = \log(-1) = \pi i + 2\pi k i \quad z_k = \frac{\pi}{2}i + \pi k i \quad k \text{ any integer}$$

$$z_0 = \frac{\pi}{2}i \quad \text{inside} \quad z_1 = \frac{3}{2}\pi i \quad \text{inside}$$

all others outside e.g. $z_{-1} = -\frac{\pi}{2}i$ $z_2 = \frac{5}{2}\pi i$.. thus

$$\int_c \frac{z}{e^z + e^{-z}} dz = 2\pi i [\operatorname{Re} s_{z_0} + \operatorname{Re} s_{z_1}] = 2\pi i \left[\frac{\frac{\pi}{2}i}{2i \sin \frac{\pi}{2}} + \frac{\frac{3\pi}{2}i}{2i \sin \frac{3\pi}{2}} \right] =$$

$$= 2\pi i \left[\frac{\pi}{4} - \frac{3}{4}\pi \right] = -\pi^2 i \quad \text{since all are simple poles and}$$

$$\operatorname{Re} s_{z_k} = \lim_{z \rightarrow z_k} \frac{z(z - z_k)}{e^z + e^{-z}} = z_k \lim_{z \rightarrow z_k} \frac{1}{e^z - e^{-z}} = \frac{z_k}{2i \sin z_k}.$$

6. Evaluate $\int_{-1}^1 z^i dz = F(1) - F(-1)$ where $F(z) = \frac{z^{1+i}}{1+i}$

we know that $z^{1+i} = e^{(1+i)\log z}$ where for $\log z$ we are using the branch with $\arg z \in \left(\frac{\pi}{2}, \frac{5}{2}\pi\right)$

$$\text{so } \log 1 = 0 + 2\pi i \quad \log(-1) = 0 + \pi i$$

$$F(1) = e^{2\pi i(1+i)} = e^{2\pi i} e^{-2\pi} = e^{-2\pi} \quad F(-1) = e^{\pi i(1+i)} = e^{\pi i} e^{-\pi} = -e^{-\pi}$$

$$\text{thus the integral } I = \frac{1}{1+i} (e^{-2\pi} + e^{-\pi}) = \frac{1-i}{2} (e^{-2\pi} + e^{-\pi}).$$

7. For singular points of $f(z) = \frac{1}{e^{z^2} - 1}$ solve $e^{z^2} = 1$ $z^2 = \log 1 = 2k\pi i$

$$k = 0 \quad z_0 = 0 \quad \text{double pole since } \lim_{z \rightarrow 0} \frac{z^2}{e^{z^2} - 1} = \lim_{z \rightarrow 0} \frac{2z}{e^{z^2} 2z} = 1 \neq 0$$

$$k > 0 \quad z^2 = 2k\pi e^{i\frac{\pi}{2}} \quad z_k = \pm \sqrt{2k\pi} e^{i\frac{\pi}{4}} = \pm \sqrt{k\pi} (1+i)$$

$$k < 0 \quad z^2 = 2|k|\pi e^{-i\frac{\pi}{2}} \quad z_k = \pm \sqrt{2|k|\pi} e^{-i\frac{\pi}{4}} = \pm \sqrt{|k|\pi} (1-i)$$

for $k \neq 0$ simple poles so

$$\operatorname{Res}_{z_k} f(z) = \left[\frac{1}{e^{z^2} 2z} \right]_{z_k} = \frac{1}{2z_k} = \frac{\pm(1-i)}{2\sqrt{k\pi}} \text{ for } k > 0$$

and $\operatorname{Re} s_{z_k} f(z) = \frac{1}{2z_k} = \frac{\pm(1+i)}{2\sqrt{|k|}\pi}$ for $k < 0$

BUT

$$\operatorname{Re} s_0 f(z) = \lim_{z \rightarrow 0} \left[\frac{z^2}{e^{z^2} - 1} \right]' = \lim_{z \rightarrow 0} \frac{2z [e^{z^2} - 1 - z^2 e^{z^2}]}{(e^{z^2} - 1)^2} = \lim_{z \rightarrow 0} 2z \cdot \lim_{w \rightarrow 0} \frac{e^w - 1 - we^w}{(e^w - 1)^2} = 0$$

since for the second one we can use L'H.Rule

$$\lim_{w \rightarrow 0} \frac{e^w - 1 - we^w}{(e^w - 1)^2} = \lim_{w \rightarrow 0} \frac{e^w - e^w - we^w}{2(e^w - 1)e^w} = \lim_{w \rightarrow 0} \frac{-w}{2(e^w - 1)} = -\frac{1}{2}$$

also in Laurent series we have only even powers so $b_1 = 0$

$$8. \text{ First } \int_0^\infty \frac{x^2 + 1}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 + 1}{x^4 + 1} dx = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2 + 1}{x^4 + 1} dx$$

we create a curve γ consisting from the line segment $[-R, R]$ and c_R = upper half of the circle

$|z| = R > 1$ oriented positively

then $\int_\gamma \frac{z^2 + 1}{z^4 + 1} dz = 2\pi i [\operatorname{Res}_{z_1} + \operatorname{Res}_{z_2}]$ where z_1 and z_2 are poles with $\operatorname{Im} z > 0$

$$\text{solve } z^4 = -1 = e^{i\pi + 2k\pi i} \quad z_k = e^{i(\frac{\pi}{4} + k\frac{\pi}{2})}$$

$$\text{for } k = 0 \quad z_1 = \frac{1}{\sqrt{2}}(1 + i) \quad z_1^2 = i$$

$$k = 1 \quad z_2 = \frac{1}{\sqrt{2}}(-1 + i) \quad z_2^2 = -i \text{ (the other two } z = \frac{1}{\sqrt{2}}(\pm 1 - i) \text{ are outside } \gamma)$$

$$\operatorname{Re} s_{z_k} = \frac{z_k^2 + 1}{4z_k^3} = \frac{z_k^2 + 1}{4z_k^4} \cdot z_k \quad z_k^4 = -1$$

$$\operatorname{Res}_{z_1} + \operatorname{Res}_{z_2} = \frac{i + 1}{-4} \cdot \frac{1}{\sqrt{2}}(1 + i) + \frac{-i + 1}{-4} \cdot \frac{1}{\sqrt{2}}(-1 + i) = \frac{i}{-2\sqrt{2}} \cdot 2 = -\frac{i}{\sqrt{2}}$$

since the degree of the bottom \geq than the degree of the top we know that

$$\int_{c_R} \frac{z^2 + 1}{z^4 + 1} dz \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\int_{c_R} \frac{z^2 + 1}{z^4 + 1} dz + \int_{-R}^R \frac{x^2 + 1}{x^4 + 1} dx = \int_\gamma \frac{z^2 + 1}{z^4 + 1} dz = 2\pi i [\operatorname{Res}_{z_1} + \operatorname{Res}_{z_2}] = \sqrt{2}\pi$$

$$\text{and finally } \int_0^\infty \frac{x^2 + 1}{x^4 + 1} dx = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2 + 1}{x^4 + 1} dx = \frac{\pi}{\sqrt{2}}$$

TOTAL out of 50: