Pmat 421
Assignment # 3 due by Monday March 10 , 4pm.

Each questions is worth 5 points.

1. $\sin z = i \qquad \frac{1}{2i}(e^{iz} - e^{-iz}) = i \to (e^{iz} - e^{-iz}) = -2$
   set $w = e^{iz}$ then $\qquad w - \frac{1}{w} = -2 \to w^2 + 2w - 1 = 0$
   so $\qquad w_{1,2} = -1 \pm \sqrt{2} \qquad$ P.V. $\qquad w = -1 + \sqrt{2}$
   thus $\qquad iz = \log\left(-1 - \sqrt{2}\right) = \ln\left(1 + \sqrt{2}\right) + i\pi + 2k\pi i$
   $z = \pi + 2k\pi - i\ln(1 + \sqrt{2}), k$ any integer
   also $iz = \log\left(-1 + \sqrt{2}\right) = \ln\left(\sqrt{2} - 1\right) + 0 + 2k\pi i$
   $z = 2k\pi - i\ln(\sqrt{2} - 1), k$ any integer
   for $k = 0 \qquad$ *P.V.* $\qquad z = -i\ln\left(\sqrt{2} - 1\right) = i\ln(\sqrt{2} + 1)$

2. For (a) $\qquad i^{-1-i} = e^{(-1-i)\log i} = e^{(-1-i)\left(i\frac{\pi}{2} + 2k\pi i\right)} = e^{-\left(i\frac{\pi}{2} + 2k\pi i\right)} e^{\frac{\pi}{2} + 2k\pi} = -ie^{\frac{\pi}{2} + 2k\pi}, k$ any integer
   (b) $\qquad (-1-i)^i = e^{i\log(-1-i)} = e^{i\left(\ln\sqrt{2} - \frac{3}{4}\pi i + 2k\pi i\right)} = e^{\frac{3}{4}\pi + 2k\pi} e^{i\ln\sqrt{2}} = e^{\frac{3}{4}\pi + 2k\pi}\left[\cos\left(\ln\sqrt{2}\right) + i\sin\left(\ln\sqrt{2}\right)\right]$

3. $\qquad \cos z = i\sin z \qquad \to \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{i}{2i}(e^{iz} - e^{-iz}) \to e^{iz} + e^{-iz} = e^{iz} - e^{-iz}$
   thus $\qquad e^{-iz} = -e^{-iz} \qquad w = -w$ only for $w = 0 \qquad$ which is NOT possible since $e^z \neq 0$ for any z.

4. For (a) $\qquad f(z) = \tan^3 z$ is defined if $\cos z \neq 0 \to z \neq \frac{\pi}{2} + k\pi$
   and by Rules $f'(z) = \left(\tan^3 z\right)' = 3\tan^2 z \cdot \sec^2 z = \dfrac{3\sin^2 z}{\cos^4 z}$
   (b) for $\; z \neq 0 \qquad f'(z) = -2\sin 2z - \frac{1}{z^2}\cos\frac{1}{z}$.
   (a) .

5. *Logw* has the cut on $\{\operatorname{Im} w = 0, \operatorname{Re} w \leq 0\}$
   since $\operatorname{Im}(3z - i) = 3y - 1$ and $\operatorname{Re}(3z - i) = 3x$
   the function is analytic on the complex plane minus the branch cut$=\{(x, y); y = \frac{1}{3}, x \leq 0\}$
   and $\; f'(z) = \dfrac{3}{3z - i}$ for all $\; z \notin \{\operatorname{Im} z = \frac{1}{3}, \operatorname{Re} z \leq 0\}$.

6. We know that $e^{\log w} = w$ for any $w \neq 0$ and for any branch of log

apply exp to $Log(z^2-1) = i\frac{\pi}{2} \to z^2 - 1 = e^{i\frac{\pi}{2}} = i$

thus $z^2 = 1+i = \sqrt{2}e^{i\left(\frac{\pi}{4}+2k\pi\right)}$ $z = \pm\sqrt[4]{2}e^{i\frac{\pi}{8}} = \pm\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

7. the limit $\lim_{z\to\infty} e^{-z}$ **does not exist**

since$|e^z| = e^x$ and real exp.$e^x \to \infty$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$

8. $\int_0^1 (1+2it)^3\, dt =$ (by antider.) $\left[\frac{(1+2it)^4}{8i}\right]_0^1 = \frac{-i}{8}\left[(1+2i)^4 - 1\right] =$

$= \frac{-i}{8}\left[(-3+4i)^2 - 1\right] = \frac{-i}{8}[-8-24i] = -3+i$

OR

$(1+2it)^3 = 1 + 6it - 12t^2 - 8it^3$ and integrate term by term...

9. $\int_0^2 \frac{t}{(t^2+i)^2} dt = \left[\frac{(t^2+i)^{-1}}{-2}\right]_0^2 = \frac{1}{2}\left[\frac{1}{i} - \frac{1}{4+i}\right] = \frac{1}{2}\left[-i - \frac{4-i}{17}\right] =$

$= -\frac{2}{17} - \frac{8}{17}i$ by Fund Th.only

10. Evaluate $\int_{-2}^0 (1+i)\cos it\ dt = (1+i)\int_{-2}^0 \cos it\ dt = (1+i)\left[\frac{\sin it}{i}\right]_{-2}^0 =$

$= (1+i)\,\frac{1}{i}\,[0 - \sin(-2i)] = (1+i)\,(-i)\,\frac{e^{-2}-e^2}{2i} = (1+i)\,\frac{e^2 - e^{-2}}{2}$

OR

use $\cos(it) = \sinh t$ $\sin(it) = i\,\sinh t$ $(\sinh t)' = \cosh t$

$\int_{-2}^0 (1+i)\cos it\ dt = (1+i)\int_{-2}^0 \cosh t\ dt = (1+i)\,[\sinh t]_{-2}^0 = (1+i)\sinh 2$

BONUS QUESTION for 10 points.:

11. Find the conditions on a function f which is analytic in a domain D such that $\mathrm{Re}(f'(z)) = 0$ for all $z \in D$.

we know that for $z = x + iy$

$f'z) = u_x + iv_x = v_y - iu_y$ $\mathrm{Re}(f'(z)) = 0$ implies

$u_x = 0 \to u = u(y)$ only $v_y = 0 \to v = v(x)$ only

and

$f'z) = iv_x = -iu_y \to v'(x) = -u'(y)$ possible only if

$v'(x) = -u'(y) \ = a \qquad$ a real constant

thus $v(x) = ax + b$ and $u(y) = -ay + c$, where b, c are real

and $\qquad f(z) = a(-y + ix) + c + ib = aiz + w_0$

where a is a real number and w_0 is a complex number.