Pmat 421
Assignment # 2 -Solution

1. Express $\quad (-\sqrt{3-i})^{99} \quad$ in the form of $a+ib, a, b$ real.

   First polar form: $-\sqrt{3}-i=2e^{-i\frac{5}{6}\pi}$ since $\theta=\arctan\frac{1}{\sqrt{3}}-\pi$

   then $(-\sqrt{3-i})^{99}=2^{99}e^{-i\pi\left(\frac{5}{6}\cdot 99\right)}=2^{99}e^{-i\frac{\pi}{2}}=-i2^{99}$

   since $\frac{5}{6}\cdot 99=\frac{5}{2}\cdot 33=\frac{165}{2}=\frac{164}{2}+\frac{1}{2}=82+\frac{1}{2}$ and $e^{-i82\pi}=1$

2. Solve $\quad z^2=7-24i \quad$ in the form of $a+ib, a, b$ real

   $(x+iy)^2=x^2-y^2+2ixy=7-24i \qquad x^2-y^2=7$ and $xy=-12$

   solve the system of equations

   $y=-\dfrac{12}{x}\to x^2-\frac{144}{x^2}=7\to x^4-7x^2-144=\left(x^2-16\right)\left(x^2+9\right)$

   thus $\quad x=\pm 4 \qquad y=\mp 3 \qquad z=\pm(4-3i)$.

3. For $w=\frac{1}{z}$ find the image of the set $S=\{z;|z|=2,\operatorname{Im} z>0\}$.

   using polar form $z=re^{i\theta}$

   $S=\{z;r=2,\theta\in(0,\pi)\}$ and $w=\frac{1}{r}e^{-i\theta}$

   $f(S)=\left\{|w|=\frac{1}{2},\arg w\in(-\pi,0)\right\}$

4. What is the best upper bund of $|z-3|$ if $z\in N_1(i)-$ the neighborhood of i with radius 1?

   $|z-3|=|z-i+i-3|\leq|z-i|+|i-3|\leq 1+\sqrt{10}$.

5. For $f(z)=e^{\frac{1}{z}}$ find the domain of definition and the functions u and v such that $f(z)=u(x,y)+iv(x,y)$ for $z=x+iy$.Is it onto C ? Explain..

   for $\quad z\neq 0 \qquad \dfrac{1}{\bar{z}}=\dfrac{x+iy}{x^2+y^2} \qquad e^{\frac{1}{z}}=e^{\operatorname{Re}\frac{1}{z}}e^{i\operatorname{Im}\frac{1}{\bar{z}z}}$

   $u(x,y)=e^{\frac{x}{x^2+y^2}}\cos\frac{y}{x^2+y^2} \qquad v(x,y)=e^{\frac{x}{x^2+y^2}}\sin\frac{y}{x^2+y^2}$

   we know that the range of exp. f. is not including 0 so it is not onto;

   generally,two numbers are missing from the range of f: $\quad$ 0 and 1

   since $e^{\#}$ is never zero and $e^0=1$ but $\frac{1}{z}$ is never 0.

6. Sketch/describe the set $\left|e^{z-\frac{1}{z}}\right|=1$.Is it open,closed, bounded,connected? Explain.

   for $z\neq 0 \qquad \left|e^{z-\frac{1}{z}}\right|=e^{\operatorname{Re}\left(z-\frac{1}{z}\right)}=1$ iff $\operatorname{Re}(z-\frac{1}{z})=0$

   $z-\frac{1}{z}=x-\dfrac{x-iy}{x^2+y^2}\to x-\dfrac{x}{x^2+y^2}=x\left(1-\dfrac{1}{x^2+y^2}\right)=0$

so $x = 0$ or $x^2 + y^2 = 1$ $z = iy, y \neq 0$ or $|z| = 1$

unit circle and y-axis without the origin

the set is connected , unbounded and closed since the boundary = the set.

7. $\lim_{z \to i} \dfrac{z^2 + iz + 2}{3 + 4iz - z^2} = \lim_{z \to i} \dfrac{(z - i)(z + 2i)}{-(z - i)(z - 3i)} = \lim_{z \to i} \dfrac{(z + 2i)}{-(z - 3i)} = \frac{3i}{2i} = \dfrac{3}{2}$.

8. for a) $\lim_{z \to 0} \dfrac{iz + \bar{z}}{|z|^2}$:

   if $z = x (y = 0)$ $\dfrac{iz + \bar{z}}{|z|^2} = \dfrac{x(1 + i)}{|x|^2} = \dfrac{1 + i}{x}$ $\left|\dfrac{1 + i}{x}\right| \to \infty$

   also $z = iy\,(x = 0)$ $\dfrac{iz + \bar{z}}{|z|^2} = \dfrac{-y\,(1 + i)}{y^2}$ $\left|\dfrac{1 + i}{-y}\right| \to \infty$

   but $z = x\,(1 + i)\,(y = x)$ $\dfrac{iz + \bar{z}}{|z|^2} = 0$ thus limit DNE-does not exist.

   for b) $\lim_{z \to \infty} \dfrac{iz + \bar{z}}{|z|^2} = 0$ since $\left|\dfrac{iz + \bar{z}}{|z|^2}\right| \leq \dfrac{|iz| + |\bar{z}|}{|z|^2} = \dfrac{2\,|z|}{|z|^2} = \dfrac{2}{|z|} \to 0$ as $|z| \to \infty$

9. Find all z where $f(z) = z^2 \bar{z}$ is differentiable, then find $f'(z)$ for such z.

   $f(z) = z^2 \bar{z}$ could be differentiable only at 0 since $z^2$is differentiable everywhere and $\bar{z}$ nowhere

   and

   $f\prime(0) = \lim_{z \to 0} \dfrac{z^2 \bar{z} - 0}{z} = \lim_{z \to 0} z\bar{z} = 0$

   OR

   $f(z) = z^2 \bar{z} = \left(x^2 - y^2 + 2ixy\right)(x - iy) \to u = x^3 + xy^2$ $v = x^2 y + y^3$

   check C.R.$u_x = 3x^2 + y^2 = v_y = x^2 + 3y^2$ and $u_y = 2xy = -v_x = -2xy$

   it must $x^2 = y^2$ and $xy = 0 \to\to (0, 0)$

   partials are cont evrywhere , C.R. are satisfied only for $z = 0$ thus f has derivative

   only at $z = 0$ and $f'(0) = u_x\,(0, 0) + iv_x\,(0, 0) = 0$

   Nowhere analytic

10. For $f(z) = \dfrac{z^2}{\bar{z}}$ for $z \neq 0$ and $f(0) = 0$. f is continous but NOT differntiable at 0

    $f'(0) = \lim_{z \to 0} \dfrac{\frac{z^2}{\bar{z}} - 0}{z} = \lim_{z \to 0} \dfrac{z}{\bar{z}}$ DNE since $\dfrac{z}{\bar{z}} = 1$ for $z = x$ and $\dfrac{z}{\bar{z}} = -1$ for $z = iy$