PMAT 315 ASSIGNMENT 3 SOLUTIONS

1. (a) Let $G \approx G'$ be isomorphic groups and let H be a subgroup of G. Prove that G' has a subgroup H' which is isomorphic to H.

(b) Page 149 #22.

1. (a) Suppose that $\phi : G \to G'$ is an isomorphism between G and G'. We prove that $\phi(H) = \{\phi(h) | h \in H\}$ is a subgroup of G' which is isomorphic to H. $\phi(H)$ is a subgroup of G', by Theorem 6.3 #4 (page 127). And the isomorphism is simply $\phi|_H$, ϕ restricted to H. Since ϕ is one-to-one, so is $\phi|_H$. Since $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$, the same will be true for $\phi|_H$ (and all $a, b \in H$). And $\phi|_H$ is obviously onto $\phi(H)$.

(b) Let g be some nonidentity element of G, and look at $<g>$, the subgroup of G generated by g. Since G has no nontrivial proper subgroups, $<g>$ must be all of G, so G must be cyclic. If G is infinite, then $G \approx \mathbb{Z}$. But $\mathbb{Z}$ has nontrivial proper subgroups, for example the subgroup E of all even integers. So by part (a), G must have a nontrivial proper subgroup isomorphic to E, which is a contradiction. Therefore G must be a finite cyclic group, so $G \approx \mathbb{Z}_n$ for some integer $n > 1$. If n is composite, say $n = st$ for integers s and t strictly between 1 and n, then $H = <s> = \{x \in \mathbb{Z}_n \mid x \text{ is a multiple of } s\}$ is a nonempty proper subgroup of $\mathbb{Z}_n$. By part (a), G must have a nonempty proper subgroup isomorphic to H, which again is a contradiction. Therefore $|G| = n$ must be prime.

2. (a) Let G be a finite group with $|G|$ odd. Prove that every element of G has odd order.

(b) Page 149 #30.

(c) Give an example of as noncyclic group of odd order.

2. (a) Let $g \in G$. Then $<g>$ is a subgroup of G, and by Lagrange's Theorem its order must divide into $|G|$ which is odd, so the order of $<g>$ (which is the same as the order of g) must also be odd.

(b) Suppose that H is a subgroup of odd order of the dihedral group D_n. Then by part (a), every element of H must have odd order. The elements of D_n are of two types: rotations (including the identity), and reflections. The order of every reflection is 2, which is even; thus H cannot contain any reflections, and so H must consist entirely of rotations. The subgroup of all rotations is a cyclic subgroup of D_n, and H is a subgroup of this subgroup, so H must be cyclic too (by Theorem 4.3 page 78).

(c) One example is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, which has order 9, but is not cyclic, by Theorem 8.2 on page 156.

3. (a) Page 165 #4.

(b) Page 166 #20.

(c) Actually, it turns out that $D_6 \approx S_3 \oplus \mathbb{Z}_2$. Prove this using the "internal direct product" theorem we did in class (page 189, Theorem 9.6 with $n = 2$). That is, find normal subgroups H and K of D_6 so that $H \approx S_3$, $K \approx \mathbb{Z}_2$, $H \cap K = \{e\}$, and $HK = D_6$. You may use Example 11 on page 66 and Example 2 on page 178.

3. (a) ($\Rightarrow$) Assume $G \oplus H$ is Abelian, where G and H are groups. We want to prove that G and H are Abelian. By symmetry it will be enough to prove that G is Abelian. So let $a, b \in G$ be arbitrary. Then we want to prove that $ab = ba$. Let $h \in H$ be arbitrary (for example we could let $h = e_H$). Then (a, h) and (b, h) are in $G \oplus H$. Since $G \oplus H$ is Abelian, $(a, h)(b, h) = (b, h)(a, h)$. But

$$(ab, h^2) = (a, h)(b, h) = (b, h)(a, h) = (ba, h^2),$$

and so by equating first coordinates we get $ab = ba$ as desired. Therefore G is Abelian, and a similar argument shows that H is Abelian.

($\Leftarrow$) Assume that G and H are Abelian groups. We want to show that $G \oplus H$ is Abelian. So we let (g_1, h_1) and (g_2, h_2) be arbitrary elements of $G \oplus H$, where $g_1, g_2 \in G$ and $h_1, h_2 \in H$, and we want to show that $(g_1, h_1)(g_2, h_2) = (g_2, h_2)(g_1, h_1)$. Since G and H are Abelian, $g_1 g_2 = g_2 g_1$ and $h_1 h_2 = h_2 h_1$. Thus

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2) = (g_2 g_1, h_2 h_1) = (g_2, h_2)(g_1, h_1).$$

Therefore $G \oplus H$ is Abelian.

(b) Since S_3 is not Abelian, we know from part (a) that $S_3 \oplus \mathbb{Z}_2$ is not Abelian. Thus, since $\mathbb{Z}_{12}$ and $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ are Abelian, we know that $S_3 \oplus \mathbb{Z}_2 \not\approx \mathbb{Z}_{12}$ and $S_3 \oplus \mathbb{Z}_2 \not\approx \mathbb{Z}_6 \oplus \mathbb{Z}_2$. Also, the element R_{120} of S_3 has order 3, and the element 1 of $\mathbb{Z}_2$ has order 2, so by Theorem 8.1 on page 155 the element $(R_{120}, 1)$ of $S_3 \oplus \mathbb{Z}_2$ has order $\text{lcm}(3, 2) = 6$. However, by Example 5 on page 142, A_4 has no element of order 6. Therefore by Theorem 6.2 #5 on page 126, $S_3 \oplus \mathbb{Z}_2 \not\approx A_4$. By elimination the only possibility is that $S_3 \oplus \mathbb{Z}_2 \approx D_6$.

(c) By Example 11 on page 66, $Z(D_6) = \{R_0, R_{180}\}$, a 2-element subgroup of D_6, and thus $Z(D_6) \approx \mathbb{Z}_2$. By Example 2 on page 178, $Z(D_6)$ is a normal subgroup of D_6. Thus we can use $K = Z(D_6)$.

Now for H. Draw a regular hexagon $ABCDEF$ (labelled counterclockwise, with vertex A at the top). Note that ACE are the vertices of an equilateral triangle. We will find six symmetries of the hexagon which correspond to the six symmetries of this equilateral triangle. First notice that the rotations R_0, R_{120}, R_{240} of the hexagon also are symmetries of the triangle ACE. For example, R_{120} (counterclockwise) moves A to C, C to E, and E to A. So these three hexagon rotations restricted to the vertices A, C, E are just the rotations of the equilateral triangle ACE. Furthermore, reflections of the hexagon about each of the three lines AD, BE, CF are reflections of the triangle; for instance, the hexagon reflection about the vertical line AD keeps A fixed and exchanges C and E. We will call this reflection V; the reflection about the line BE (which has negative slope) will be called N, and the reflection about the line CF (which has positive slope) will be called P. Thus these three hexagon reflections restricted to the vertices A, C, E are just the three reflections of the triangle ACE. So the six symmetries $R_0, R_{120}, R_{240}, V, P, N$ must form a subgroup of D_6 isomorphic to D_3. Since $D_3 \approx S_3$, we can put $H = \{R_0, R_{120}, R_{240}, V, P, N\}$ and then $H \approx S_3$. Since $|H| = 6 = |D_6|/2$ and using a result we saw in class (or Exercise 7 page 193), H is a normal subgroup of D_6.

Notice that $H \cap K = \{R_0\}$, the identity. So we only need to show that $HK = D_6$. But it is clear that every element in $H \cup K$ will be in HK, because the identity R_0 is in both H and K. Also $R_{60} = R_{240}R_{180} \in HK$ and $R_{300} = R_{120}R_{180} \in HK$. So we need only show that the three reflections of the hexagon which are not in H (these are the reflections about the lines joining midpoints of opposite sides of the hexagon) are in HK. By symmetry we need only show one of these is in HK. But $VR_{180} \in HK$ is such a reflection:

$$VR_{180}(A) = V(D) = D; \quad VR_{180}(B) = V(E) = C; \quad VR_{180}(C) = V(F) = B;$$

$$VR_{180}(D) = V(A) = A; \quad VR_{180}(E) = V(B) = F; \quad VR_{180}(F) = V(C) = E.$$

This symmetry switches A and D, B and C, and E and F, so it is the reflection about the line joining the midpoints of sides BC and EF. So we are done.

4. (a) Page 193 #40.

(b) Give an example of an abelian group G (with $|G| > 1$) and a nontrivial proper subgroup H of G so that every element of H and every element of G/H is a square.

4. (a) We assume that G is an Abelian group, H is a subgroup of G (and thus a normal subgroup since G is Abelian), every element of H is a square, and every element of G/H is a square. We want to prove that every element of G is a square. So let $g \in G$ be arbitrary. Then $gH \in G/H$, so by assumption $gH = (g_1H)^2 = g_1^2H$ for some $g_1 \in G$. Since $g \in gH$, $g \in g_1^2H$, which says that $g = g_1^2h$ for some $h \in H$. By assumption, $h = h_1^2$ for some $h_1 \in H$. Thus $g = g_1^2h_1^2 = (g_1h_1)^2$ since G is Abelian, so g is a square.

(b) One example is $G = (\mathbb{R}, +)$ and $H = \mathbb{Q}$. Then every element of H is a square means (using additive notation) that every rational number q can be written as $q_1 + q_1 = 2q_1$ for some $q_1 \in \mathbb{Q}$, which of course is true because $q_1 = q/2$ is rational. Also, every element of G/H is a square means that every left coset $r + \mathbb{Q}$ in $\mathbb{R}/\mathbb{Q}$ can be written in the form $(r_1 + \mathbb{Q}) + (r_1 + \mathbb{Q}) = 2r_1 + \mathbb{Q}$, which is also true because we can put $r_1 = r/2$ for any $r \in \mathbb{R}$.

5. (a) Page 194 #49. (See pages 32–33 for the notation.)

(b) Page 194 #53.

5. (a) You can assume without proof that the given K and L are both subgroups of D_4. Since $|D_4| = 8$ and $|L| = 4 = |D_4|/2$, we know from class (or Exercise 7 page 193) that $L \lhd D_4$. Since $|L| = 4$ and $|K| = 2$, we know for the same reason that $K \lhd L$. But from the table on page 33 we get for example that

$$R_{90}DR_{90}^{-1} = R_{90}DR_{270} = HR_{270} = D' \notin K,$$

so (since $D \in K$) $R_{90}KR_{90}^{-1} \not\subseteq K$, so K is not normal in D_4.

(b) Let $N \lhd G$ where N is cyclic. Let H be an arbitrary subgroup of N. We want to prove that $H \lhd G$. So let $g \in G$ be arbitrary, and we want to prove that $gHg^{-1} \subseteq H$. To do this, let $h \in H$ be arbitrary, and we want to prove that $ghg^{-1} \in H$. Since N is cyclic, N is generated by some element $n \in N$. Since $h \in H$ which is a subgroup of N, $h \in N$, so $h = n^k$ for some integer k. Thus $ghg^{-1} = gn^kg^{-1} = (gng^{-1})^k$. Since $N \lhd G$, $gng^{-1} \in gNg^{-1} \subseteq N$, so $gng^{-1} = n^t$ for some integer t since N is generated by n. Thus $ghg^{-1} = (n^t)^k = (n^k)^t = h^t \in H$.