

The University of Calgary
 Faculty of Science
 Department of Mathematics and Statistics
 MATH 353 -02 Midterm Test

Time:90 minutes

Winter 2008

Each questions is for 10 marks.

1. For $f(x, y) = 3x - x^3 - xy^2$ find all local extrema .

$$\text{solve } \nabla f = 0 \quad f_x = 3 - 3x^2 - y^2 = 0 \quad f_y = -2xy = 0$$

from the second equation two cases:

$$x = 0 \rightarrow y^2 = 3 \quad (0, \pm\sqrt{3})$$

$$y = 0 \rightarrow 3x^2 = 3 \quad (\pm 1, 0)$$

second der. Test:

$$f_{xx} = -6x \quad f_{xy} = -2y \quad f_{yy} = -2x$$

$$\text{and } D = 4y^2 - 12x^2 \quad \text{for } (0, \pm\sqrt{3}) \quad D > 0 \quad \text{saddle points}$$

$$\text{for } (1, 0) \quad D < 0, A < 0 \quad \text{local maximum}$$

$$\text{for } (-1, 0) \quad D < 0, A > 0 \quad \text{local minimum}$$

2. For $f(x, y) = 3x - x^3 - xy^2$ find the absolute extrema on the triangle with vertices $(0, 0)$, $(2, 2)$ and $(2, 0)$.

$$\text{the set } T = \{0 \leq x \leq 2, 0 \leq y \leq x\}$$

using problem 1: inside the triangle no points

and on the boundary $(1, 0)$

generally 3 parts of the **boundary**: $B_1 = \{0 \leq x \leq 2, 0 = y\}$

$$\text{then } f \text{ on } B_1 = g(x) = 3x - x^3 \quad g'(x) = 3(1 - x^2) = 0 \quad (1, 0)$$

and ends $(0, 0)$ and $(2, 0)$

$B_2 = \{x = 2, 0 \leq y \leq 2\}$ then f on $B_2 = h(y) = -2(1 + y^2)$ only ends $(2, 2)$

$B_3 = \{0 \leq x \leq 2, y = x\}$ then f on $B_3 = l(x) = 3x - 2x^3$

$$l'(x) = 3 - 6x^2 = 0 \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

now values: $f(1, 0) = 2 \dots$ maximum

$$f(0, 0) = 0 \quad f(2, 0) = -2 \quad f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$f(2, 2) = -10 \dots \text{minimum}$$

$$3. \iint_D \frac{x}{1-xy} dx dy = \int_0^1 \left(\int_0^1 \frac{x}{1-xy} dy \right) dx = (u = 1 - xy, du = -x dy) =$$

$$\begin{aligned}
&= \int_0^1 [-\ln(1-xy)]_{y=0}^{y=1} dx = \int_0^1 -\ln(1-x) dx = (u = 1-x, du = -dx) = \\
&= \int_1^0 \ln u du = -[u \ln u - u]_0^1 = 1
\end{aligned}$$

it is convergent since

$$\lim_{u \rightarrow 0^+} u \ln u = "0 \cdot -\infty" \quad \lim_{u \rightarrow 0^+} \frac{\ln u}{u^{-1}} = (\text{L'H.R.}) \lim_{u \rightarrow 0^+} \frac{u^{-1}}{-u^{-2}} = \lim_{u \rightarrow 0^+} -u = 0$$

4. in the coordinates x, y, z

- (a) if in $S = \{\rho = 2, \phi = \frac{3}{4}\pi, \theta \in [0, 2\pi)\} \rightarrow z = 2 \cos \frac{3}{4}\pi = -\sqrt{2}$
and $x = 2 \cos \theta \sin \frac{3}{4}\pi \quad y = 2 \sin \theta \sin \frac{3}{4}\pi$ so
 $x = \sqrt{2} \cos \theta \quad y = \sqrt{2} \sin \theta$
circle $x^2 + y^2 = 2$, radius $=\sqrt{2}$ on the horiz. plane $z = -\sqrt{2}$
- (b) if in $S = \{z = 2, r \geq 0, \theta = \frac{\pi}{2}\} \quad x = 0, z = 2, y = r \geq 0$
 $\cos \frac{\pi}{2} = 0, \sin \frac{\pi}{2} = 1 \quad (0, r, 2)$
a ray = half line parallel to the y - axis in the yz - plane

5. By cyl.coord. $\iiint_B \frac{xz}{\sqrt{x^2+y^2}} dx dy dz = I$

where $B = \{(x, y, z); x^2 + y^2 + z^2 \leq 2, z \geq 0, x \geq 0, x^2 + y^2 \geq 1\}$

inside the sphere ,outside the cylinder

$$I = \iiint_{B^*} \frac{r \cos \theta \cdot z}{r} r dr d\theta dz \text{ where } B^* = \{(r, \theta, z); r^2 + z^2 \leq 2, z \geq 0, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], r \geq 1\}$$

thus $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta \int_{D^*} r z dr dz$ where $D^* = \{(r, z); r^2 + z^2 \leq 2, z \geq 0, r \geq 1\} = \{1 \leq r \leq \sqrt{2}, 0 \leq z \leq \sqrt{2-r^2}\}$

$$\text{so } I = 2 [\sin \theta]_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \left(r \int_0^{\sqrt{2-r^2}} z dz \right) dr = 2 \int_1^{\sqrt{2}} r \left[\frac{z^2}{2} \right]_0^{\sqrt{2-r^2}} dr = \int_1^{\sqrt{2}} r (2-r^2) dr = \left[r^2 - \frac{r^4}{4} \right]_1^{\sqrt{2}} =$$

$$2 - 1 - 1 + \frac{1}{4} = \frac{1}{4}$$

also : $0 \leq z \leq 1, 1 \leq r \leq \sqrt{2-z^2}$

6. from $\int_0^1 \left(\int_z^1 \left(\int_0^x e^{x^3} dy \right) dx \right) dz$ given: $0 \leq z \leq 1, z \leq x \leq 1, 0 \leq y \leq x$

outside x: $0 \leq x \leq 1$ and then $0 \leq y \leq x, 0 \leq z \leq x$ thus

$$\int_0^1 \left(\int_z^1 \left(\int_0^x e^{x^3} dy \right) dx \right) dz = \int_0^1 e^{x^3} \left(\int_0^x \left(\int_0^x dy \right) dz \right) dx = \int_0^1 x^2 e^{x^3} dx = \left[\frac{1}{3} e^{x^3} \right]_0^1 = \frac{1}{3} (e - 1).$$

7. the integral $I = \iiint_B y \, dx dy dz$ where $B = \{(x, y, z); x^2 + y^2 + z^2 \leq 4, z \geq 0, y \geq 0\}$

in cylindrical $I = \iiint_{B^*} r \sin \theta \cdot r \, dr d\theta dz$ where $B^* = \{r^2 + z^2 \leq 4, z \geq 0, \theta \in [0, \pi]\}$

so
$$I = \int_0^\pi \sin \theta d\theta \int_0^2 r^2 \left(\int_0^{\sqrt{4-r^2}} dz \right) dr = [-\cos \theta]_0^\pi \cdot \int_0^2 r^2 \sqrt{4-r^2} dr \text{ (Table)...}$$

in spherical coordinates $I = \iiint_{B^{**}} \rho \sin \theta \sin \phi \cdot \rho^2 \sin \phi \, d\rho d\theta d\phi$

where $B^{**} = \{0 \leq \rho \leq 2, \phi \in [0, \frac{\pi}{2}], \theta \in [0, \pi]\}$

thus

$$I = \int_0^\pi \sin \theta d\theta \int_0^{\frac{\pi}{2}} \sin^2 \phi \, d\phi \int_0^2 \rho^3 d\rho = 2 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\phi}{2} d\phi \left[\frac{\rho^4}{4} \right]_0^2 = \left[\frac{\pi}{2} - 0 \right] \cdot 4 = 2\pi.$$