

The University of Calgary  
 Faculty of Science  
 Department of Mathematics and Statistics  
 MATH 353  
 Midterm Supplement

1. **for a)**  $\rho = x \rightarrow \sqrt{x^2 + y^2 + z^2} = x$

so  $x \geq 0$  and  $x^2 + y^2 + z^2 = x^2, y^2 + z^2 = 0$

thus  $y = z = 0$  and any point  $(x, 0, 0)$  for  $x \geq 0$  - half of the x-axis

**for b)**  $\rho = -2y \rightarrow \sqrt{x^2 + y^2 + z^2} = -2y$

so  $y \leq 0$  and  $x^2 + y^2 + z^2 = 4y^2$

$x^2 + z^2 = 3y^2$ , one part cone around the y - axis  $y = -\frac{\sqrt{x^2 + z^2}}{\sqrt{3}}$ .

**for a)**

first  $B = \{(x, y, z); x \geq 0, y \geq 0, z \geq 0, \frac{3x + 2y}{6} \leq z \leq 2\}$

"bottom" must be below "top"  $\frac{3x + 2y}{6} \leq 2$  and we get

$D = \{(x, y); x \geq 0, y \geq 0, 3x + 2y \leq 12\}$  a triangle

so 
$$\iiint_B z \, dx dy dz = \iint_D \left( \int_{\frac{3x+2y}{6}}^2 z \, dz \right) dx dy = \int_0^4 \left[ \int_0^{\frac{12-3x}{2}} \left( \int_{\frac{3x+2y}{6}}^2 z \, dz \right) dy \right] dx = ..$$

**for b)**

$0 \leq z \leq 2$  and for a fixed  $z$

$D_z = \{(x, y); x \geq 0, y \geq 0, 3x + 2y \leq 6z\}$

triangles with the base  $[0, 2z]$  and the height  $[0, 3z]$  and the area  $3z^2$

$$\iiint_B z \, dx dy dz = \int_0^2 z \left( \iint_{D_z} dx dy \right) dz = \int_0^2 z (\text{area of } D_z) dz = 3 \int_0^2 z^3 dz = 12$$

OR

$$\iiint_B z \, dx dy dz = \int_0^2 z \left( \int_0^{2z} \left( \int_0^{\frac{6z-3x}{2}} dy \right) dx \right) dz = ...$$

$B = \{(x, y, z); x^2 + y^2 + z^2 \leq 4; x^2 + y^2 \geq 3, x \geq 0, y \geq 0\}$

**for a)**

the solid  $B$  is outside the cylinder and inside the sphere

$x \geq 0, y \geq 0$  implies  $\theta \in [0, \frac{\pi}{2}]$

$x^2 + y^2 + z^2 \leq 4 \rightarrow 0 \leq \rho \leq 2$

and  $x^2 + y^2 \geq 3$  implies  $\rho \sin \phi \geq \sqrt{3}$   $\frac{\sqrt{3}}{\sin \phi} \leq \rho \leq 2$

so necessarily  $\frac{\sqrt{3}}{2} \leq \sin \phi$   $\phi \in [\frac{\pi}{3}, \frac{2\pi}{3}]$

$$B^* = \left\{ \theta \in \left[0, \frac{\pi}{2}\right], \phi \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right], \frac{\sqrt{3}}{\sin \phi} \leq \rho \leq 2 \right\}$$

as iterated integrals

$$\begin{aligned} \iiint_B \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}} &= \iiint_{B^*} \frac{\rho^2 \sin \phi d\rho d\phi d\theta}{\rho} = \frac{\pi}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin \phi \left( \int_{\frac{\sqrt{3}}{\sin \phi}}^2 \rho d\rho \right) d\phi = \\ &= \frac{\pi}{4} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin \phi \left( 4 - \frac{3}{\sin^2 \phi} \right) d\phi = \pi \left[ -\cos \phi \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} - \frac{3\pi}{4} \left[ \ln |\csc \phi - \cot \phi| \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} = \end{aligned}$$

$$\begin{aligned} &\left( \cos \frac{\pi}{3} = \frac{1}{2}, \cos \frac{2\pi}{3} = -\frac{1}{2}, \sin(\text{both}) = \frac{\sqrt{3}}{2}, \cot = \pm \frac{1}{\sqrt{3}} \right) \\ &= \pi - \frac{3\pi}{2} \ln \sqrt{3} = \pi \left( 1 - \frac{3}{4} \ln 3 \right) \end{aligned}$$

also from the symmetry  $I = 2 \cdot (\text{integral for } z > 0) \rightarrow 2 \cdot \left\{ \phi \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \right\}$

**for b)**

the solid  $B$  is outside the cylinder and inside the sphere

$$x \geq 0, y \geq 0 \text{ implies } \theta \in \left[0, \frac{\pi}{2}\right], x^2 + y^2 + z^2 \leq 4 \rightarrow r^2 + z^2 \leq 4$$

$$\text{and } x^2 + y^2 \geq 3 \text{ implies } r \geq \sqrt{3}$$

$$B^* = \left\{ \theta \in \left[0, \frac{\pi}{2}\right], r \geq \sqrt{3}, r^2 + z^2 \leq 4 \right\}$$

$$\iiint_B \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}} = \iiint_{B^*} \frac{r dr dz d\theta}{\sqrt{r^2 + z^2}} = \frac{\pi}{2} \int_{\sqrt{3}}^2 r \left( \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \frac{dz}{\sqrt{r^2 + z^2}} \right) dr =$$

$$\text{OR} = \frac{\pi}{2} \int_{-1}^1 \left( \int_{\sqrt{3}}^{\sqrt{4-z^2}} \frac{r dr}{\sqrt{r^2 + z^2}} \right) dz = \pi \int_0^1 \left( \sqrt{r^2 + z^2} \right)_{\sqrt{3}}^{\sqrt{4-z^2}} dz = \pi \int_0^1 \left( 2 - \sqrt{3 + z^2} \right) dz =$$

$$\begin{aligned} &= 2\pi - \frac{\pi}{2} \left[ z\sqrt{3 + z^2} + 3 \ln \left| z + \sqrt{3 + z^2} \right| \right]_0^1 = 2\pi - \frac{\pi}{2} [2 + 3 \ln 3 - 3 \ln \sqrt{3}] = \\ &= \pi \left[ 1 - \frac{3}{4} \ln 3 \right]. \end{aligned}$$

the solid is a tetrahedron in the first octant under the plane  $3x + 3y + z = 6$

$$\text{so } B = \{x \geq 0, y \geq 0, z \geq 0, 3x + 3y + z \leq 6\}$$

for  $z = 0$  we have a triangle  $(0, 0), (2, 0), (0, 2)$  also  $D_0$

for  $z = 6$   $x = y = 0$  point  $(0, 0, 6)$

for  $z \in (0, 6)$  we have a smaller triangle

under the line  $3x + 3y = 6 - z$  with vertices  $(0, 0), (0, \frac{6-z}{3}), (\frac{6-z}{3}, 0)$

$$D_z = \{x \geq 0, y \geq 0, x + y \leq \frac{6-z}{3}\} \dots \text{triangle with area } A_z = \frac{1}{2} \left( \frac{6-z}{3} \right)^2$$

$$\iiint_B \frac{dx dy dz}{z-6} = \int_0^6 \frac{1}{z-6} \left( \iint_{D_z} dx dy \right) dz = \int_0^6 \frac{1}{z-6} \cdot A_z dz = \frac{1}{18} \int_0^6 (z-6) dz =$$

$$= \frac{1}{36} [(z-6)^2]_0^6 = \frac{-36}{36} = -1.$$

2.  $B = \{x^2 + y^2 + z^2 \leq 2, z \geq 0, y \geq 0, x^2 + y^2 = z\}$

for a)

the set is inside the sphere with radius  $\sqrt{2}$

below the paraboloid  $z = x^2 + y^2$ , and above the xy-plane  $z = 0$

$$B^* = \{r^2 + z^2 \leq 2, z \geq 0, r \geq 0, \theta \in [0, \pi], r^2 \geq z\}$$

$$\text{so } I = \iiint_B z \, dx dy dz = \iiint_{B^*} z \, r dr d\theta dz = \pi \iint_D r z \, dr dz$$

$$\text{where } D = \{r^2 + z^2 \leq 2, z \geq 0, r \geq 0, r^2 \geq z\}$$

the intersection of the circle and parabola is at  $z = r = 1$

it is easier to slice it horizontally in  $rz$ - plane

$$\text{so } z \in [0, 1] \text{ and } r \geq \sqrt{z}, r \leq \sqrt{2 - z^2}$$

$$\text{the integral } I = \pi \int_0^1 z \left( \int_{\sqrt{z}}^{\sqrt{2-z^2}} r dr \right) dz = \pi \int_0^1 z \left( \left[ \frac{r^2}{2} \right]_{\sqrt{z}}^{\sqrt{2-z^2}} \right) dz =$$

$$= \frac{1}{2} \pi \int_0^1 z ([2 - z^2 - z]) dz = \frac{\pi}{2} \left[ z^2 - \frac{z^4}{4} - \frac{z^3}{3} \right]_0^1 = \frac{\pi}{2} \cdot \left( 1 - \frac{7}{12} \right) = \frac{5}{24} \pi.$$

for b)

$$x^2 + y^2 + z^2 \leq 2 \rightarrow \rho \leq \sqrt{2}$$

$$z \geq 0 \rightarrow \phi \in [0, \frac{\pi}{2}] \quad y \geq 0 \rightarrow \theta \in [0, \pi]$$

$$x^2 + y^2 \geq z \rightarrow \rho^2 \sin^2 \phi \geq \rho \cos \phi \rightarrow \rho \geq \frac{\cos \phi}{\sin^2 \phi}$$

$$\text{but necessarily } \sqrt{2} \geq \frac{\cos \phi}{\sin^2 \phi} \rightarrow \phi \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right]$$

thus

$$B^* = \left\{ \rho \leq \sqrt{2}, \theta \in [0, \pi], \phi \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right], \rho \geq \frac{\cos \phi}{\sin^2 \phi} \right\}$$

and

$$I = \iiint_B z \, dx dy dz = \iiint_{B^*} \rho^3 \cos \phi \sin \phi \, d\rho d\phi d\theta =$$

$$= \int_0^\pi d\theta \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left( \cos \phi \sin \phi \int_{\frac{\cos \phi}{\sin^2 \phi}}^{\sqrt{2}} \rho^3 d\rho \right) d\phi \dots$$

$$B = \{x^2 + y^2 \leq z \leq 4 - x^2 - 3y^2\} \text{ between two paraboloids}$$

$$\text{necessarily } x^2 + y^2 \leq 4 - x^2 - 3y^2$$

$$\text{so } 2x^2 + 4y^2 \leq 4 \text{ that is } (x, y) \in D_0 = \{x^2 + 2y^2 \leq 2\} \text{ -ellipse}$$

and

$$\begin{aligned} \iiint_B \sqrt{x^2 + 2y^2} dx dy dz &= \iint_{D_0} \sqrt{x^2 + 2y^2} \left( \int_{x^2+y^2}^{4-x^2-3y^2} dz \right) dx dy = \\ &= \iint_{D_0} \sqrt{x^2 + 2y^2} (4 - 2(x^2 + 2y^2)) dx dy \end{aligned}$$

use modified polar coord. to evaluate

$$x = r \cos \theta, y = \frac{1}{\sqrt{2}} r \sin \theta$$

$$\text{then } x^2 + 2y^2 = r^2 \quad dx dy = \frac{1}{\sqrt{2}} r dr d\theta$$

$$\text{so } D_0 \text{ transforms into } D^* = \{0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi\}$$

and the integral into

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (4r - 2r^3) \frac{r}{\sqrt{2}} dr = \sqrt{2}\pi \left[ \frac{4r^3}{3} - \frac{2r^5}{5} \right]_0^{\sqrt{2}} = \frac{32}{15}\pi.$$

$$B = \{x \geq 0, y \geq 0, z \geq 0, z \leq 2 - y, x \leq 4 - y^2 \text{ ( or } y \leq \sqrt{4 - x} \text{)}\}$$

so

$$D_0 = \{x \geq 0, y \geq 0, x \leq 4 - y^2 \text{ ( or } y \leq \sqrt{4 - x} \text{)}\}$$

**and for a)**

$$\iiint_B f dx dy dz = \iint_{D_0} \left( \int_0^{2-y} f dz \right) dx dy$$

**for b)**

$$0 \leq z \leq 2 \text{ and for a fixed } z \quad D_z = \{0 \leq y \leq 2 - z, 0 \leq x \leq 4 - y^2\}$$

$$\iiint_B f dx dy dz = \int_0^2 \left( \iint_{D_z} f dx dy \right) dz$$

