

MATH 353
Handout #5 Solution

For 1)

let's find the intersection of two given surfaces

$$z = x^2 + y^2 \rightarrow z + z^2 = 2 \text{ so } z^2 + z - 2 = 0 \text{ has positive sol. } z = 1$$

and we can describe the surface

$$S = \{z = \sqrt{2 - x^2 - y^2}, (x, y) \in D\} \text{ where } D = \{x^2 + y^2 \leq 1\}$$

$$\text{and normal } \mathbf{n} = \pm (\nabla z, -1) = \pm \left(\frac{x}{\sqrt{2-x^2-y^2}}, \frac{y}{\sqrt{2-x^2-y^2}}, 1 \right)$$

$$\text{and } \|\mathbf{n}\| = \sqrt{\frac{2}{2-x^2-y^2}} k, \text{ finally the surface area}$$

$$A = \iint_S dS = \iint_D \|\mathbf{n}\| dx dy = \sqrt{2} \iint_D \frac{dx dy}{\sqrt{2-x^2-y^2}} = (\text{polar})$$

$$= 2\pi\sqrt{2} \int_0^1 \frac{r dr}{\sqrt{2-r^2}} = 2\sqrt{2}\pi \left[-\sqrt{2-r^2} \right]_0^1 = 2\pi \left[2 - \sqrt{2} \right].$$

For 2)

the field on the lateral part of $S_l = \{x^2 + y^2 = 4, 0 \leq z \leq 3\}$

is $\mathbf{F} = (1, 1, 16z)$

but since this part is vertical $x^2 + y^2 = 4, z \in [0, 3]$

we cannot describe it as $z = f(x, y)$ so parametrize $\mathbf{r}(u, v)$:

$$x = 2 \cos u \quad y = 2 \sin u \quad z = v \quad u \in [0, 2\pi], v \in [0, 3]$$

$$\frac{\partial \mathbf{r}}{\partial u} = (-2 \sin u, 2 \cos u, 0) \quad \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1) \text{ and the normal vector}$$

$$\mathbf{n} = \pm \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) = (2 \cos u, 2 \sin u, 0) \quad (\|\mathbf{n}\| = 2)$$

for outward take +

$$\mathbf{F} \cdot \mathbf{n} = 2(\cos u + \sin u)$$

$$\iint_{S_l} \mathbf{F} \cdot d\mathbf{S} = 2 \int_0^{2\pi} \int_0^3 (\cos u + \sin u) dv du = 0 \text{ because of the periodicity.}$$

The top could be described as $z = 3$

$(x, y) \in D = \{x^2 + y^2 \leq 4\}$ $\mathbf{n} = (0, 0, 1)$ upward

the field on the top is $\mathbf{F} = (1, 1, 3(x^2 + y^2)^2)$ and $\mathbf{F} \cdot \mathbf{n} = 3(x^2 + y^2)^2$

the flux coming out from the top is

$$\iint_{top} \mathbf{F} \cdot d\mathbf{S} = \iint_D 3(x^2 + y^2)^2 dx dy = (\text{polar}) = 6\pi \int_0^2 r^5 dr = 2^6 \pi.$$

The bottom could be described as $z = 0$

$(x, y) \in D = \{x^2 + y^2 \leq 4\}$ $\mathbf{n} = (0, 0, -1)$ downward

the field on the bottom is $\mathbf{F} = (1, 1, 0)$ and $\mathbf{F} \cdot \mathbf{n} = 0$.

Therefore the only contribution to the flux is out from the top and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 192\pi.$$

For 3 a)

the surface S is vertical so parametrize:

$$x = 2 \cos u, y = 2 \sin u, u \in \left[0, \frac{\pi}{2}\right] \quad z = v$$

since $0 \leq z \leq 5 - 2x - y$ (below the plane)

necessary condition $2x + y \leq 5$ is satisfied inside the circle

$$\mathbf{r}(u, v) = (2 \cos u, 2 \sin u, v) \text{ and } D = \left\{u \in \left[0, \frac{\pi}{2}\right], 0 \leq v \leq 5 - 4 \cos u - 2 \sin u\right\}$$

$$\frac{\partial \mathbf{r}}{\partial u} = (-2 \sin u, 2 \cos u, 0), \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1) \text{ so } \mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (2 \cos u, 2 \sin u, 0)$$

$$\text{thus } \|\mathbf{n}\| = 2 \text{ and } SA = \iint_S dS = \iint_D \|\mathbf{n}\| \, dudv = 2 \int_0^{\frac{\pi}{2}} \left(\int_0^{5-4\cos u-2\sin u} dv \right) du =$$

$$= 2 \int_0^{\frac{\pi}{2}} (5 - 4 \cos u - 2 \sin u) du = 5\pi + [-8 \sin u + 4 \cos u]_0^{\frac{\pi}{2}} = 5\pi - 8 - 4 = 5\pi - 12.$$

For 3 b)

this time S is given as $z = 5 - 2x - y, (x, y) \in D = \{x^2 + y^2 \leq 4\}$

then $\mathbf{n} = (-2, -1, -1)$ and $\|\mathbf{n}\| = \sqrt{6}$

$$SA = \iint_S dS = \iint_D \|\mathbf{n}\| \, dxdy = \sqrt{6} \text{area of } D = 4\sqrt{6}\pi.$$

For 4)

$S : z = \sqrt{4 - y^2}, (x, y) \in D = \{x^2 + y^2 \leq 4\}$ since $z > 0$

normal vector $\mathbf{n} = (-\nabla z, 1) = \left(0, \frac{y}{\sqrt{4 - y^2}}, 1\right)$

since upward means positive z-coordinate

$\mathbf{F} = (x^2yz, y, xz)$ on S $\mathbf{F} = (\dots, y, x\sqrt{4 - y^2})$ and

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \mathbf{n} \, dxdy = \iint_D \left[\frac{y^2}{\sqrt{4 - y^2}} + x\sqrt{4 - y^2} \right] dxdy = \\ &= \int_{-2}^2 \left(\frac{y^2}{\sqrt{4 - y^2}} \int_{-\sqrt{4 - y^2}}^{\sqrt{4 - y^2}} dx \right) dy + \int_{-2}^2 \sqrt{4 - y^2} \left(\int_{-\sqrt{4 - y^2}}^{\sqrt{4 - y^2}} x dx \right) dy = \\ &= 2 \int_0^2 \frac{y^2}{\sqrt{4 - y^2}} \cdot 2\sqrt{4 - y^2} dy + 0 (x \text{ is an odd f}) = 4 \int_0^2 y^2 dy = \frac{4}{3} \cdot 8 = \frac{32}{3}. \end{aligned}$$

For 5)

$S : z = \frac{x^2}{2}, (x, y) \in D = \{x^2 + y^2 \leq 1, x > 0, y < 0\}$

normal vector $\mathbf{n} = (\nabla z, -1) = (x, 0, -1)$ and $\|\mathbf{n}\| = \sqrt{x^2 + 1}$

and

$$\begin{aligned} \iint_S zx \, dS &= \iint_D \frac{x^2}{2} \cdot x\sqrt{x^2 + 1} \, dxdy = \frac{1}{2} \int_0^1 x^3 \sqrt{x^2 + 1} \left(\int_{-\sqrt{1 - x^2}}^0 dy \right) dx = \\ &= \frac{1}{2} \int_0^1 x^3 \sqrt{x^2 + 1} \sqrt{1 - x^2} dx = \frac{1}{2} \int_0^1 x^3 \sqrt{1 - x^4} dx = \\ (u = 1 - x^4, du = -4x^3 dx) &= \frac{1}{8} \int_0^1 \sqrt{u} du = \frac{1}{8} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^1 = \frac{1}{12}. \end{aligned}$$

For 6)

Evaluate where $S : z = 2 - x - y$ for $(x, y) \in D = \{x^2 + 2y^2 \leq 1\}$

since for $z = 0$ the line $x + y = 2$ is outside the ellipse

so $\mathbf{n} = (\nabla z, -1) = (-1, -1, -1)$ and $\|\mathbf{n}\| = \sqrt{3}$

and $\iint_S x^2 dS = \int_D \int x^2 \sqrt{3} dx dy$ (modified polar . or cartesian coord.)

$$x = r \cos \theta \quad y = \frac{1}{\sqrt{2}} r \sin \theta \quad dx dy = \frac{1}{\sqrt{2}} r dr d\theta \quad x^2 + 2y^2 = r^2$$

then $D^* = \{0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ and the integral

$$= \sqrt{\frac{3}{2}} \int_0^1 \int_0^{2\pi} r^2 \cos^2 \theta r d\theta dr = \sqrt{\frac{3}{2}} \left[\frac{r^4}{4} \right]_0^1 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta = \frac{\sqrt{3}\pi}{4\sqrt{2}}$$

$$\text{OR} \quad x \in [-1, 1] \text{ and } y \in \left[-\frac{1}{\sqrt{2}} \sqrt{1-x^2}, \frac{1}{\sqrt{2}} \sqrt{1-x^2} \right].$$

For 7)

For the vector field $\mathbf{F}(x, y, z) = (xz, 4xyz^2, 2z)$ calculate

$$\text{div} \mathbf{F} = (xz)_x + (4xyz^2)_y + (2z)_z = z + 4xz^2 + 2$$

and where

$$\text{curl } \mathbf{F} = \begin{vmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ xz & 4xyz^2 & 2z \end{vmatrix} = (-8xyz, x, 4yz^2).$$

For 8)

Laplace operator : $f_{xx} + f_{yy} = 0$

$$\text{so } f_{xx} = (2Axy)_x = 2Ay = -f_{yy} = -(3By^2)_y = -6By$$

thus any B and $A = -3B$ will do.