

**The University of Calgary**  
**Department of Mathematics and Statistics**  
**MATH 353      Handout #4      Solution**

**1. For 1a)**

$$\text{Given } F_1 = 3x^2yz \quad F_2 = kyz + x^3z \quad F_3 = x^3y + 1 + y^2).$$

$$\text{necessary condition } (F_1)_y = 3x^2z = (F_2)_x = 3x^2z$$

$$(F_1)_z = 3x^2y = (F_3)_x = 3x^2y, \text{ and finally}$$

$$(F_2)_z = ky + x^3 = (F_3)_y = x^3 + 2y \text{ gives us } k = 2.$$

**For 1b)**

$$f_x = F_1 = 3x^2yz \quad \text{so by integrating with respect to } x :$$

$$f(x, y, z) = \int (3x^2yz \, dx + c(y, z)) = x^3yz + c(y, z)$$

$$\text{differentiate } f_y = F_2 = 2yz + x^3z = x^3z + \frac{\partial c}{\partial y} \text{ thus } \frac{\partial c}{\partial y} = 2yz \text{ and } c(y, z) = y^2z + c(z)$$

$$\text{together } f(x, y, z) = x^3yz + y^2z + c(z)$$

$$\text{differentiate } f_z = F_3 = x^3y + 1 + y^2 = x^3y + y^2 + c'(z) \text{ thus } c' = 1 \text{ and } c(z) = z + c$$

$$\text{and finally the general potential } f(x, y, z) = x^3yz + y^2z + z + c \text{ where } c \text{ is any constant.}$$

**For 2)**

$$\text{line of intersection of two planes } x + y - z = 1 \text{ and } 2x + y - 3z = 0$$

$$\text{first let's find a direction vector } \mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 \text{ (of normal vectors of the given planes)}$$

$$\mathbf{d} = (1, 1, -1) \times (2, 1, -3) = (-2, 1, -1) \text{ so}$$

$$(x, y, z) = (3, 0, 2) + t(-2, 1, -1) \text{ and when } t = 0 \text{ we get } D$$

$$\text{OR } x = 3 - 2t \quad y = t \quad z = 2 - t$$

$$\text{to get a point on the } xy\text{-plane } z = 0 \text{ so } t = 2 \text{ and the point is } P(-1, 2, 0)$$

$$\text{Together } \mathbf{r}(t) = (3 - 2t, t, 2 - t) \quad t \in [0, 2]$$

$$\mathbf{r}'(t) = (-2, 1, -1) = \mathbf{d} \quad \|\mathbf{r}'(t)\| = \sqrt{6}$$

$$\text{the given function } f(z) = z^2 \text{ evaluated on } c \quad f \circ \mathbf{r} = (2 - t)^2$$

$$\int_c f \, ds = \int_0^2 (2 - t)^2 \|\mathbf{r}'(t)\| \, dt = \sqrt{6} \left[ \frac{(t - 2)^3}{3} \right]_0^2 = \frac{8}{3} \sqrt{6}.$$

**For 3)**

$$F_1 = ky^2 + x \text{ and } F_2 = xy - \frac{1}{\sqrt{y}} \text{ for } y > 0$$

$$(F_1)_y = 2ky = (F_2)_x = y \text{ so } k = \frac{1}{2}$$

then

$$f_x = \frac{1}{2}y^2 + x \text{ and } f_y = xy - \frac{1}{\sqrt{y}}$$

$$f = \int f_x dx = \int \left(\frac{1}{2}y^2 + x\right) dx + c(y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 + c(y)$$

$$f_y = xy + c'(y) = F_2 = xy - \frac{1}{\sqrt{y}} \quad c'(y) = -\frac{1}{\sqrt{y}} \text{ for } y > 0$$

$$\text{and } c(y) = -2\sqrt{y} \quad \text{together } f(x, y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 - 2\sqrt{y} + \text{const.}$$

**For 4)**

intersection of  $z - y = 1$  and  $0 = x - y^2$  between  $A(1, -1, 0)$  and  $B(0, 0, 1)$

if  $y = t$  then  $x = y^2, z = 1 + y$  so  $\mathbf{r}(t) = (t^2, t, 1 + t)$  for  $t \in [-1, 0]$

$$\mathbf{r}'(t) = (2t, 1, 1) \text{ and } \|\mathbf{r}'(t)\| = \sqrt{2 + 4t^2}$$

$$\int_c z ds = \int_{-1}^0 (1+t)\sqrt{2+4t^2} dt = \int_{-1}^0 \sqrt{2+4t^2} dt + \int_{-1}^0 t\sqrt{2+4t^2} dt =$$

(for the first integral:  $u = 2t, a = \sqrt{2}, dt = \frac{1}{2}du$  then Table for  $\sqrt{2+u^2}$ )

OR  $\sqrt{2+4t^2} = 2\sqrt{\frac{1}{2}+t^2}, a = \frac{1}{\sqrt{2}}$  Table; for the second one  $u = 2+4t^2$ )

$$\begin{aligned} &= \frac{1}{2} \left[ t\sqrt{2+4t^2} + \ln(2t + \sqrt{2+4t^2}) \right]_{-1}^0 + \left[ \frac{(2+4t^2)^{\frac{3}{2}}}{12} \right]_{-1}^0 = \\ &= \frac{\sqrt{6}}{2} + \frac{1}{2} \ln \sqrt{2} - \frac{1}{2} \ln(\sqrt{6} - 2) + \left[ \frac{\sqrt{2}}{6} - \frac{\sqrt{6}}{2} \right] = \sqrt{\frac{3}{2}} - \frac{1}{2} \ln(\sqrt{3} - \sqrt{2}). \end{aligned}$$

**For 5)**

$$\mathbf{r}(t) = (t, t^2, e^t), t \in [1, 2] \quad \mathbf{r}'(t) = (1, 2t, e^t)$$

then the field on  $c: \mathbf{F} \circ \mathbf{r} = (e^t, e^t, 2t)$  for  $\mathbf{F}(x, y, z) = (z, e^{\frac{y}{x}}, 2x)$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_1^2 \mathbf{F} \cdot \mathbf{r}' dt = \int_1^2 (e^t + 4te^t) dt = (\text{second one by parts})$$

$$= [e^t + 4te^t - 4e^t]_1^2 = 5e^2 - e.$$

**For 6)**

$F_1 = (3x\sqrt{x^2+y^4} + \cos x)$  and  $F_2 = ky^3\sqrt{x^2+y^4} + \sin y$  for any point except the origin

$$(F_1)_y = \frac{6xy^3}{\sqrt{x^2+y^4}} = (F_2)_x = \frac{kxy^3}{\sqrt{x^2+y^4}} \text{ so } k = 6 \text{ then } f = ?$$

$$f_x = (3x\sqrt{x^2+y^4} + \cos x) \text{ and } f_y = 6y^3\sqrt{x^2+y^4} + \sin y$$

$$f = \int f_x dx = \int (3x\sqrt{x^2+y^4} + \cos x) dx + c(y) = (x^2+y^4)^{\frac{3}{2}} + \sin x + c(y)$$

$$f_y = \frac{3}{2}\sqrt{x^2+y^4} \cdot 4y^3 + c'(y) = F_2 = 6y^3\sqrt{x^2+y^4} + \sin y$$

$$c'(y) = \sin y \quad \text{and} \quad c(y) = -\cos y$$

together  $f(x, y) = (x^2+y^4)^{\frac{3}{2}} + \sin x - \cos y + \text{const.}$

**For 7)**

$$\mathbf{r}(t) = (t \cos t, t \sin t, t), t \in [0, 1] \quad \mathbf{r}'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$$

$$\|\mathbf{r}'(t)\| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} = \sqrt{2 + t^2}$$

$$\int_c z \, ds = \int_0^1 (t\sqrt{2+t^2}) dt = \left[ \frac{(2+t^2)^{\frac{3}{2}}}{3} \right]_0^1 = \sqrt{3} - \frac{2\sqrt{2}}{3}.$$

**for 8)**

intersection of the plane  $z = 2x$  and the paraboloid  $z = x^2 + y^2 \rightarrow 2x = x^2 + y^2$

$$1 = (x - 1)^2 + y^2 \quad x = 1 + \cos t \quad y = \sin t \quad \text{and} \quad z = 2x$$

$$\mathbf{r}(t) = (1 + \cos t, \sin t, 2 + 2 \cos t), t \in [0, 2\pi]$$

$$\mathbf{r}'(t) = (-\sin t, \cos t, -2 \sin t)$$

then the field on  $c : \mathbf{F} \circ \mathbf{r} = (\sin t, 2 + 2 \cos t, 0)$  for  $\mathbf{F} = (y, z, 2x - z)$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{2\pi} (-\sin^2 t + 2 \cos t + 2 \cos^2 t) dt =$$

( using  $-\sin^2 t = -1 + \cos^2 t$  )

$$= -2\pi + [2 \sin t]_0^{2\pi} + \frac{3}{2} \int_0^{2\pi} (1 + \cos 2t) dt = -2\pi + 0 + 3\pi = \pi.$$