

The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #3 Solution

For 1)

when $z = 0$ $\cos \sqrt{x^2 + y^2} = 0$ $\sqrt{x^2 + y^2} = \frac{\pi}{2} + k\pi$, we can choose $k = 0$

so $D = \{(x, y); x^2 + y^2 \leq \left(\frac{\pi}{2}\right)^2\}$ a circular disk

and $V = \iint_D \cos \sqrt{x^2 + y^2} dx dy$ (polar coord.) $= \iint_{D^*} r \cos r dr d\theta$

where $D^* = \{(r, \theta); 0 < r \leq \frac{\pi}{2}; 0 \leq \theta < 2\pi\}$ so by parts

$$V = 2\pi \int_0^{\frac{\pi}{2}} r \cos r dr = 2\pi \left[r \sin r - \int \sin r de \right]_0^{\frac{\pi}{2}} = \pi^2 + 2\pi [\cos r]_0^{\frac{\pi}{2}} = \pi^2 - 2\pi.$$

For 2)

using polar coord. $\iint_D e^{3(x^2+y^2)} dx dy = \iint_{D^*} e^{3r^2} r dr d\theta = \pi \left[\frac{e^{3r^2}}{6} \right]_1^2 = \frac{\pi}{6} (e^{12} - e^3)$

where $D^* = \{0 \leq \theta \leq \pi, 1 \leq r \leq 2\}$ region between two circles, only top half

For 3)

$T = \{0 \leq x \leq 2, 2x \leq y \leq 4\}$ so

$$I = \iint_T \frac{1}{(y-2x)^k} dA = \int_0^2 \left(\int_{2x}^4 (y-2x)^{-k} dy \right) dx$$

$$\text{recall } \int_0^1 u^{-k} du = \begin{cases} k = 1 & [\ln u]_0^1 = \infty \\ k > 1 & \left[\frac{u^{1-k}}{1-k} \right]_0^1 = \infty \\ k < 1 & \left[\frac{u^{1-k}}{1-k} \right]_0^1 = \frac{1}{1-k} \end{cases}$$

similarly for the inside integral $\int_0^2 \left[\frac{(y-2x)^{1-k}}{1-k} \right]_{y=2x}^{y=4} dx$

for $k \neq 1$ and $\lim_{y \rightarrow 2x} (y-2x)^{1-k} = 0$ and finite only for $1-k > 0$

for $k = 1$ $[\ln(y-2x)]_{y=2x}^{y=4} = +\infty$

so **for only $k < 1$ the integral is convergent** and

$$I = \int_0^2 \left[\frac{(4-2x)^{1-k}}{1-k} \right] dx = \left[\frac{(4-2x)^{2-k}}{(-2)(1-k)(2-k)} \right]_0^2 = \frac{2}{(1-k)(2-k)} \text{ since } 2-k > 0$$

For 4)

the region is in the first quadrant inside the circle $r = \sqrt{2}$, on the right of vert line $x = 1$
 intersection of the line $x = 1$ and the circle is at the point $(1,1) \rightarrow \theta = \frac{\pi}{4}$

use polar coord.: $x \geq 1 \rightarrow r \cos \theta \geq 1$
 $D^* = \{0 \leq r \leq \sqrt{2}, \theta \in [0, \frac{\pi}{4}], r \geq \frac{1}{\cos \theta}\}$

necessary cond. from $\frac{1}{\cos \theta} \leq r \leq \sqrt{2}$ is $\cos \theta \geq \frac{1}{\sqrt{2}}$ so $\theta \in [0, \frac{\pi}{4}]$

$$\iint_D \frac{dx dy}{\sqrt{x^2 + y^2}} = \iint_{D^*} \frac{r dr d\theta}{\sqrt{r^2}} = \int_0^{\frac{\pi}{4}} \left(\int_{\frac{1}{\cos \theta}}^{\sqrt{2}} dr \right) d\theta = \int_0^{\frac{\pi}{4}} \left(\sqrt{2} - \frac{1}{\cos \theta} \right) d\theta =$$

$$(\text{Table}) = \frac{\pi\sqrt{2}}{4} - [\ln |\sec \theta + \tan \theta|]_0^{\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4} - [\ln |\sqrt{2} + 1| - \ln 1] =$$

$$= \frac{\pi\sqrt{2}}{4} - \ln(\sqrt{2} + 1)$$

For 5)

using polar coord.: $y = x \rightarrow \theta = \frac{\pi}{4}$
 $x, y \geq 0 \rightarrow \theta \leq \frac{\pi}{2}$

$$\iint_D \frac{x \sin \pi (x^2 + y^2)}{\sqrt{x^2 + y^2}} dx dy = \iint_{D^*} \frac{r \cos \theta \sin(\pi r^2)}{r} r dr d\theta =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos \theta d\theta \cdot \int_0^1 r \sin \pi r^2 dr = [\sin \theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \left[\frac{-\cos \pi r^2}{2\pi} \right]_0^1 = \left(1 - \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\pi}\right).$$

where $D^* = \left\{ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, 0 < r \leq 1 \right\}$

For 6)

$D = \left\{ x \in (1, +\infty), 0 \leq y \leq \frac{1}{x^2} \right\}$ is unbounded

$$I = \iint_D e^{-x^2 y} dA = \int_1^{\infty} \left(\int_0^{\frac{1}{x^2}} e^{-x^2 y} dy \right) dx = \int_1^{\infty} \left[\frac{e^{-x^2 y}}{-x^2} \right]_{y=0}^{y=\frac{1}{x^2}} dx = \int_1^{\infty} \left[\frac{e^{-1} - 1}{-x^2} \right] dx =$$

$$= \left(\frac{1}{e} - 1 \right) \left[\frac{1}{x} \right]_{x=1}^{x \rightarrow \infty} = 1 - \frac{1}{e}.$$

For 7a)

the region is inside the circle with $r = \sqrt{2}$ above the horizontal line $y = 1$
the intersection of the circle and the line $y = 1$ is at $x = \pm 1$

$$\text{so } \iint_D (x^2 + y^2) dx dy = \int_{-1}^1 \left(\int_1^{\sqrt{2-x^2}} (x^2 + y^2) dy \right) dx =$$

$$\int_{-1}^1 \left(x^2 \sqrt{2-x^2} + \frac{1}{3} (2-x^2)^{\frac{3}{2}} - x^2 - \frac{1}{3} \right) dy dx = \dots$$

For b)

in polar coord.: $(1, 1) \rightarrow \theta = \frac{\pi}{4}$ and $(-1, 1) \rightarrow \theta = \frac{3}{4}\pi$

$$\iint_D (x^2 + y^2) dx dy = \iint_{D^*} r^3 dr d\theta$$

where $D^* = \left\{ 0 < r \leq \sqrt{2}, r \geq \frac{1}{\sin \theta}, \theta \in \left[\frac{\pi}{4}, \frac{3}{4}\pi \right] \right\}$

since $y \geq 1$ implies $r \sin \theta \geq 1$ so $\sin \theta$ must be positive and $\sqrt{2} \geq \frac{1}{\sin \theta}$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\int_{\frac{1}{\sin \theta}}^{\sqrt{2}} r^3 dr \right) d\theta = \frac{1}{4} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 - \csc^4 \theta) d\theta = \frac{\pi}{2} - \frac{2}{4} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 + \cot^2 \theta) \csc^2 \theta d\theta =$$

using symmetry ($\pm x$) and $1 + \cot^2 \theta = \csc^2 \theta$ then

subst. $u = \cot \theta \quad du = -\csc^2 \theta d\theta$

$$= \frac{\pi}{2} - \frac{1}{2} \int_0^1 (1 + u^2) du = \frac{\pi}{2} - \frac{2}{3}.$$

For 8)

$D = \{x \in (1, +\infty), 0 \leq y \leq x\}$ is unbounded

$$I = \iint_D e^{-x-y} dA = \int_1^{\infty} e^{-x} \left(\int_0^x e^{-y} dy \right) dx = \int_1^{\infty} e^{-x} [-e^{-y}]_{y=0}^{y=x} dx = \int_1^{\infty} [e^{-x} - e^{-2x}] dx = \\ = \left[-e^{-x} + \frac{1}{2}e^{-2x} \right]_{x=1}^{x \rightarrow \infty} = \frac{1}{e} - \frac{1}{2e^2} \quad ("e^{-\infty}" = 0)$$

For 9a)

the region is inside the circle with $r = 1$ and above the line $x + y = 1$

in the first quadrant

intersection at $(1, 0)$ and $(0, 1)$

so
$$\iint_D \frac{1}{(x^2 + y^2)^2} dx dy = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} \frac{1}{(x^2 + y^2)^2} dy \right) dx = \text{difficult}$$

For b)

in polar coord.: $x + y \geq 1 \rightarrow r(\cos \theta + \sin \theta) \geq 1$

$$\iint_D \frac{1}{(x^2 + y^2)^2} dx dy = \iint_{D^*} r^{-3} dr d\theta$$

where $D^* = \{0 < r \leq 1, r \geq \frac{1}{\cos \theta + \sin \theta}, \theta \in [0, \frac{\pi}{2}]\}$

$$= \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1}{\cos \theta + \sin \theta}}^1 r^{-3} dr \right) d\theta = -\frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - (\cos \theta + \sin \theta)^2) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta =$$

(using $\sin 2\theta = 2 \sin \theta \cos \theta$) $= \frac{1}{4} [-\cos 2\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}$.

For 10)

the function is unbounded

the set is the region between both axes, line $y = 1$ and $x = e^y \Leftrightarrow y = \ln x$

$$I = \iint_D \frac{1 + \ln x}{y} dA = \int_0^1 \frac{1}{y} \left(\int_0^{e^y} (1 + \ln x) dx \right) dy = \int_0^1 \frac{1}{y} [x \ln x]_0^{e^y} dy = \int_0^1 \frac{1}{y} (e^y y) dy \\ = e - 1 \quad \left(\lim_{x \rightarrow 0^+} x \ln x = 0 \right)$$

if we switch the order of integration we have to split the set into

$D_1 = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $D_2 = \{1 \leq x \leq e, \ln x \leq y \leq 1\}$

then

$$.I = \iint_D \frac{1 + \ln x}{y} dA = \int_0^1 \frac{1}{y} dy \cdot \int_0^1 (1 + \ln x) dx + \int_1^e (1 + \ln x) \left(\int_{\ln x}^1 \frac{1}{y} dy \right)$$

much harder..

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