

For 1) $f(x, y) = x^2(y + 1)^3 + y^2$
 $f_x = 2x(y + 1)^3 = 0$ $f_y = 3x^2(y + 1)^2 + 2y = 0$

if $x = 0$ then $y = 0$; if $y = -1$ then No sol.

the only C.P. is $(0, 0)$ Second.der.test

$$\begin{aligned} f_{xx} &= 2(y + 1)^3 & A &= 2 \\ f_{xy} &= 6x(y + 1)^2 & B &= 0 \\ f_{yy} &= 6x^2(y + 1) + 2 & C &= 2 & D &= -4 \\ D < 0, A > 0 & & \text{loc. minimum} \end{aligned}$$

For abs. min. we have to show that $f(x, y) \geq f(0, 0) = 0$ for all (x, y)

BUT $f(R, -2) = -R^2 + 4 < 0$ for big R

thus it is NOT abs. min.

For 2)

$$\int_{-1}^0 \left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4 + 1} \right) dy \quad \text{given: } -1 \leq y \leq 0 \quad -1 \leq x \leq \sqrt[3]{y}$$

change the order $-1 \leq x \leq 0$ $x^3 \leq y \leq 0$

then

$$\begin{aligned} \int_{-1}^0 \left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4 + 1} \right) dy &= \int_{-1}^0 \left(\frac{1}{x^4 + 1} \int_{x^3}^0 dy \right) dx = \\ &= \int_{-1}^0 \frac{-x^3 dx}{x^4 + 1} = \frac{1}{4} [-\ln(x^4 + 1)]_{-1}^0 = \frac{\ln 2}{4}. \end{aligned}$$

For 3)

the distance to $P(1, 2, 2)$ is $\sqrt{(x - 1)^2 + (y - 2)^2 + (z - 2)^2}$

so looking for minimum of

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2 \text{ with } g(x, y, z) = x^2 + y^2 + z^2 = 36$$

$$\nabla f = \lambda \nabla g \quad 2(x - 1) = \lambda 2x \quad 2(y - 2) = \lambda 2y \quad 2(z - 2) = \lambda 2z$$

$$\text{gives } \lambda = \frac{x - 1}{x} = \frac{y - 2}{y} = \frac{z - 2}{z} \text{ since } \lambda xyz \neq 0$$

$$\text{thus from } \frac{y - 2}{y} = \frac{z - 2}{z} \rightarrow z = y$$

$$\text{and from } \frac{x - 1}{x} = \frac{y - 2}{y} \rightarrow xy - y = yx - 2x \quad y = 2x$$

back to the sphere

$$x^2 + 4x^2 + 4x^2 = 9x^2 = 36 \quad x = \pm 2 \text{ and we have 2 C.P.}$$

for $(-2, -4, -4)$ the distance is $\sqrt{9 + 36 + 36} = 9$ and for

$(2, 4, 4)$ is **the closest point** since distance is $\sqrt{1 + 4 + 4} = 3$ (minimum)

For 4 a)

$$D = \{ x + y \geq 2 \text{ and } x^2 + y^2 \leq 4 \} = \{ 0 \leq x \leq 2, 2 - x \leq y \leq \sqrt{4 - x^2} \}$$

so

$$\iint_D \frac{x + y}{x^2 + y^2} dx dy = \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{x + y}{x^2 + y^2} dy \right) dx = \text{OR} = \int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} \frac{x + y}{x^2 + y^2} dx \right) dy$$

$$= \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{2y}{x^2+y^2} dy \right) dx = \int_0^2 [\ln(x^2+y^2)]_{2-x}^{\sqrt{4-x^2}} dx = 2 \ln 4 - \int_0^2 \ln(2x^2 - 4x + 4) dx$$

by parts and partial fractions..

for b)

$$D^* = \{r(\cos\theta + \sin\theta) \geq 2 \text{ and } 0 \leq r \leq 2, \theta \in [0, \frac{\pi}{2}]\}$$

$$\iint_D \frac{x+y}{x^2+y^2} dx dy = \iint_{D^*} \frac{r(\cos\theta + \sin\theta)}{r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(\int_{\frac{2}{\cos\theta + \sin\theta}}^2 dr \right) d\theta =$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(2 - \frac{2}{\cos\theta + \sin\theta} \right) d\theta = \int_0^{\frac{\pi}{2}} 2(\cos\theta + \sin\theta) d\theta - 2 \cdot \frac{\pi}{2} =$$

$$= 2 [\sin\theta - \cos\theta]_0^{\frac{\pi}{2}} - \pi = 4 - \pi.$$

For 5)

the set B is a layer of a full paraboloid

use cylindrical coordinates

$$B^* = \{1 \leq z \leq 4 \text{ and } z \geq r^2, \theta \in [0, 2\pi]\}$$

$$\text{so } \iiint_B \frac{z dV}{\sqrt{x^2+y^2}} = \iiint_{B^*} \frac{z}{\sqrt{r^2}} r dr dz d\theta = 2\pi \iint_{D^*} z dr dz$$

$$\text{where } D^* = \{1 \leq z \leq 4, 0 \leq r \leq \sqrt{z}\}$$

$$\text{the integral } I = 2\pi \int_1^4 z \left(\int_0^{\sqrt{z}} dr \right) dz = 2\pi \int_1^4 z^{\frac{3}{2}} dz = \frac{4\pi}{5} [z^{\frac{5}{2}}]_1^4 = \frac{4\pi}{5} [2^5 - 1] = \frac{124\pi}{5}.$$

For 6)

$$S = \{ z = \sqrt{3x^2 + 3y^2} \quad x + z \leq 4 \}$$

$$\text{so } 0 \leq \sqrt{3x^2 + 3y^2} \leq 4 - x \quad \rightarrow \quad D = ?$$

$$SA = \iint_S dS = \iint_D \|\mathbf{n}\| dx dy$$

$$\mathbf{n} = (\nabla z, -1) = \left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}, \frac{\sqrt{3}y}{\sqrt{x^2+y^2}}, -1 \right) \text{ or } \left(\frac{3x}{\sqrt{3x^2+3y^2}}, \frac{3y}{\sqrt{3x^2+3y^2}}, -1 \right)$$

$$\text{and } \|\mathbf{n}\|^2 = \frac{3x^2 + 3y^2}{x^2 + y^2} + 1 = 4 \quad \|\mathbf{n}\| = 2$$

we have to find D the intersection of the given cone and plane

$$4 - x \geq \sqrt{3x^2 + 3y^2} \quad \text{for } x \leq 4 \quad 16 - 8x + x^2 = 3x^2 + 3y^2$$

$$16 = 2x^2 + 8x + 3y^2 \quad 24 = 2(x+2)^2 + 3y^2$$

$$\text{finally } D = \{2(x+2)^2 + 3y^2 \leq 24\} = \left\{ \left(\frac{x+2}{\sqrt{12}} \right)^2 + \left(\frac{y}{\sqrt{8}} \right)^2 \leq 1 \right\}$$

so

$$SA = 2 \cdot \text{area of an ellipse} = 2\pi ab = 2\pi \sqrt{12} \sqrt{8} = 8\pi \sqrt{6}.$$

For 7)

the curve is closed, oriented clockwise, $n = 2$ use Green's theorem

$$\oint_c \mathbf{F} \cdot d\mathbf{s} = - \iint_T [(F_2)_x - (F_1)_y] dx dy \text{ because of the orientation of } c$$

$$\text{where } T = \{0 \leq x \leq 2, 2-x \leq y \leq 2\} = \{0 \leq y \leq 2, 2-y \leq x \leq 2\}$$

and
$$\begin{aligned} & [(F_2)_x - (F_1)_y] = \pi \cos \pi x - 3xy^2 \\ & = - \iint_T [\pi \cos \pi x - 3xy^2] dx dy = \int_0^2 3y^2 \left(\int_{2-y}^2 x dx \right) dy - \int_0^2 \pi \cos \pi x \left(\int_{2-x}^2 dy \right) dx = \\ & = \int_0^2 3y^2 \cdot \frac{2^2 - (2-y)^2}{2} dy - \int_0^2 x \cdot \pi \cos \pi x dx = \frac{3}{2} \int_0^2 (4y^3 - y^4) dy - [x \sin \pi x]_0^2 - \left[\frac{\cos \pi x}{\pi} \right]_0^2 = \\ & = \frac{3}{2} \left[2^4 - \frac{2^5}{5} \right] - 0 = \frac{72}{5}. \end{aligned}$$

For 8)

by Product Rule

$$\begin{aligned} \operatorname{div}(\phi \mathbf{F}) &= (\phi F_1)_x + (\phi F_2)_y = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 = \\ &= \phi(F_1)_x + \phi(F_2)_y + (\phi_x, \phi_y) \bullet (F_1, F_2) = \phi \operatorname{div} \mathbf{F} + \operatorname{grad} \phi \bullet \mathbf{F} \end{aligned}$$

.For 9)

since c is closed, $n = 3$, we can use Stokes's Theorem

$$I = \oint_c \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \quad \mathbf{S} = ?$$

where

$$\operatorname{curl} \mathbf{F} = \left\| \begin{array}{ccc} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2 + y & y^3 - x & z^4 \end{array} \right\| = (0, 0, -2)$$

and

S is a part of the plane $z = 2 - 2x + 3y$ so $\mathbf{n} = (-\nabla z, 1) = (2, -3, 1)$

inside the cylinder $\{x^2 + y^2 \leq 4\} = D \quad \operatorname{curl} \mathbf{F} \cdot \mathbf{n} = -2$

thus
$$I = \iint_D (-2) dx dy = -2 \cdot \text{area of } D = -8\pi.$$

For 10)

since S is a closed surface use Divergence theorem $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$

and

$$I = \iint_S \mathbf{F} \cdot d\mathbf{S} = 2 \iiint_B (x + y + z) dx dy dz \quad \text{where } B = \{x^2 + y^2 + 4(z - 1)^2 \leq 4\}$$

ellipsoid symmetrical in x and in y so $I = 0 + 0 + 2 \iiint_B z dx dy dz =$

projection on the z -axis : $mz \in [0, 2]$ so
$$I = 2 \int_0^2 z \iint_{D_z} dx dy$$

where $D_z = \{x^2 + y^2 \leq 4 - 4(z - 1)^2 = 8z - 4z^2\}$ circular disc for each fixed z
with area $= \pi(8z - 4z^2)$

thus
$$I = 2 \int_0^2 z \pi [8z - 4z^2] dz = 8\pi \int_0^2 (2z^2 - z^3) dz = 8\pi \left[\frac{2}{3} \cdot 2^3 - \frac{2^4}{4} \right] = \frac{32}{3}\pi$$

OR cylindrical coordinates
$$2 \iiint_B z dx dy dz = 4\pi \iint_{D^*} z r dr dz$$

where $D^* = \{r^2 + 4(z - 1)^2 \leq 4\}$

the integral = $4\pi \int_0^2 z \left(\int_0^{\sqrt{4-4(z-1)^2}} r dr \right) dz = 8\pi \int_0^2 z(2z - z^2) dz = \dots$

Or modified spherical coord. $z = 1 + 2\rho \cos \phi$

ALSO

$$I = 2 \iiint_B z dx dy dz = 2 \iiint_B (z - 1) dx dy dz + 2 \text{volume of } B$$

the ellipsoid: $B = \{x^2 + y^2 + 4(z - 1)^2 \leq 4\} = \left\{ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + (z - 1)^2 \leq 1 \right\}$

$$= 0 + 2 \cdot \frac{4}{3} \pi abc = \frac{32}{3} \pi.$$

For 11)

the curve is not closed but the field is conservative

so we can find a potential $F = \nabla f$

$$f_x = y \quad f_y = x \quad f_z = 2z$$

$$f(x, y, z) = xy + z^2 \text{ and } \int_c \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A) = 5 - 3 = 2$$

OR find a parametrization of c

from the cylinder $x = \sqrt{2} \cos t, y = \sqrt{2} \sin t$ and from $z = 2xy$

we have $\mathbf{r}(t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 4 \cos t \sin t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 2 \sin 2t)$

now for A $t = \frac{3}{4}\pi$, and for B $t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = (-\sqrt{2} \sin t, \sqrt{2} \cos t, 4 \cos 2t) \text{ and the field on } c$$

$$\mathbf{F} = (y, x, 2z)_c = (\sqrt{2} \sin t, \sqrt{2} \cos t, 4 \sin 2t) \text{ then}$$

$$\mathbf{F} \cdot \mathbf{r}' = -2 \sin^2 t + 2 \cos^2 t + 8 \sin 4t$$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \mathbf{F} \cdot \mathbf{r}' dt = - \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} [2 \cos 2t + 8 \sin 4t] dt =$$

$$= [-\sin 2t + 2 \cos 4t]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 1 + 1 - 2 + 2 = 2.$$

For 12)

$$\mathbf{F}(x, y, z) = (\arctan z(x^2 + y^2), \ln(1 + y^2 + z^2), y e^{xyz})$$

for any (x, y, z)

$$\text{div} \mathbf{F} = \frac{2xz}{1 + z^2(x^2 + y^2)^2} + \frac{2y}{1 + y^2 + z^2} + xy^2 e^{xyz}$$

$$\text{and } \text{curl} \mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arctan z(x^2 + y^2) & \ln(1 + y^2 + z^2) & ye^{xyz} \end{bmatrix} =$$

$$= \left(e^{xyz} (1 + xyz) - \frac{2z}{1 + y^2 + z^2}, -y^2 z e^{xyz} + \frac{x^2 + y^2}{1 + z^2(x^2 + y^2)^2}, -\frac{2yz}{1 + z^2(x^2 + y^2)^2} \right).$$