

## Practice Problems S3

1. Determine whether the following matrices are elementary matrices or not; write down the inverses of the elementary matrices (explain your answer):

$$(a) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, (b) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, (c) \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (d) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(e) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, (f) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

2. Find an invertible matrix  $U$  such that the product  $R = UA$  is the reduced row-echelon form of  $A$  if

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{bmatrix}.$$

3. Express the following matrix as a product of elementary matrices:

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$

4. Find the matrix of the reflection in the line  $y = -x$ .
5. Find a rotation or a reflection that is equal to
- (a) reflection in the  $y$ -axis followed by rotation through  $\pi/2$ ;
  - (b) rotation through  $\pi/2$  followed by reflection in the line  $y = x$ .

6. Given  $T([1 \ -2]^T) = [3 \ 4]^T$  and  $T([-2 \ 5]^T) = [-1 \ 4]^T$ , find  $T([-4 \ 3]^T)$  if  $T$  is a linear transformation.
7. Consider a Markov chain that starts in state 1 with transition matrix  $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ .
- (a) Explain why the chain is regular.
  - (b) Find the probability that the chain is in state 1 after 2 transitions.
  - (c) Find the steady-state vector for the chain.

**Recommended Problems:**

Pages 68 - 69: 1; 2a, b; 3a; 5a, b; 6 a,b; 7; 8b, c;

Pages 80-81: 1. b, c; 2. a; 3, 4, 5, 9, 10, 12; Pages 101-102: 1, 2, a, c;

Page 101-102: 1. a, b, f, g, h, k, l, m, n, o, p; 5. a, b; 6, 7, 8, 9, 11, 13, 14, 15;

## Solutions

1. Exercise.
2. To find such an invertible matrix  $U$  that transforms  $A$  into its reduced row-echelon form  $R$ , we have to bring  $[A|I_3]$  to  $[R|U]$ :

$$[A|I_3] = \left[ \begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 3 & -2 & 1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 3 & 0 & 0 & 1 \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & 4 & 8 & 1 & 0 & 1 \end{array} \right]$$

(Row 2 has been replaced by row2-3xrow1, and Row 3 by  $r_3 + r_1$ )

$$r_3 \text{ by } \frac{r_3}{4} \longrightarrow \left[ \begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array} \right]$$

$$r_1 \text{ by } r_1 - 3(r_3) \text{ and } r_2 \text{ by } r_2 + 8(r_3) \longrightarrow \left[ \begin{array}{cccc|ccc} \mathbf{1} & -1 & 0 & -1 & 1/4 & 0 & -3/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array} \right]$$

$$r_1 \text{ by } r_1 + r_2 \longrightarrow \left[ \begin{array}{cccc|ccc} \mathbf{1} & 0 & 0 & -2 & -3/4 & 1 & 5/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array} \right].$$

So,  $R = \left[ \begin{array}{cccc} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \end{array} \right]$  and  $U = \left[ \begin{array}{ccc} -3/4 & 1 & 5/4 \\ -1 & 1 & 2 \\ 1/4 & 0 & 1/4 \end{array} \right]$ . One can

check that  $R = UA$ . Note that this invertible matrix  $U$  is not unique. It depends on the sequence of row operations performed to bring  $A$  to its reduced row-echelon form.

3. The matrix  $A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$  has determinant  $\det(A) = 5 \times 1 - 2 \times 3 = -1 \neq 0$ . So,  $A$  is invertible. It can be carried to the identity matrix  $I_2$  by row operations.

Step 1: Subtract 2 times row 2 from row 1:

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \left( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right)$$

Step 2: Subtract 2 times row 1 from row 2:

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \left( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \right)$$

Step 3: Multiply row 2 by -1:

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \left( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

Step 4: Subtract row 2 from row 1:

$$\longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \left( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

We have  $A^{-1} = F_4 F_3 F_2 F_1$ . Therefore,  $A = E_1 E_2 E_3 E_4$ , where  $E_1, E_2, E_3$  and  $E_4$  are elementary matrices given by  $E_1 = F_1^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ ,  $E_2 = F_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ,  $E_3 = F_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $E_4 = F_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

4. Denote by  $\vec{v}'$  the reflection of  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$  in the line  $y = x$  and  $\vec{v}''$  the reflection of  $\vec{v}'$  in the line  $y = -x$ . We know that  $\vec{v}' = \begin{bmatrix} y \\ x \end{bmatrix}$ . From the figure, we see that  $\vec{v}''$  is the reflection of  $\vec{v}'$  about the origin. Therefore,  $\vec{v}'' = \begin{bmatrix} -y \\ -x \end{bmatrix}$ . It follows that the reflection about the line  $y = -x$  is a linear transformation with matrix  $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ . It is the composition of the reflection in the line  $y = x$  with the reflection about the origin:  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ .

Figure 1: Reflection of vectors of  $\mathbb{R}^2$  about the line  $y = -x$ :

5. (a) The reflection of vectors of  $\mathbb{R}^2$  in the  $y$ -axis is defined by  $T([x \ y]^T) = [-x \ y]^T$  and the rotation  $R_{\pi/2}$  is given by  $R_{\pi/2}([x \ y]^T) = [-y \ x]$ . We have:  $R_{\pi/2} \circ T([x \ y]^T) = R_{\pi/2}(T([x \ y]^T)) = R_{\pi/2}([-x \ y]^T) = [-y \ -x]^T$ . Therefore, the reflection in the  $y$ -axis followed by the rotation through  $\pi/2$  is the reflection about the line  $y = x$ .
- (b)  $R_{\pi/2}([x \ y]^T)$  and  $T([x \ y]^T) = [y \ x]^T$  is the reflection about the line  $y = x$ . We have  $T \circ R_{\pi/2}([x \ y]^T) = T(R_{\pi/2}([x \ y]^T)) = T([-y \ x]^T) = [x \ -y]^T$ . It is the reflection in the  $x$ -axis.
6. Given that  $T([1 \ -2]^T) = [3 \ 4]^T$  and  $T([-2 \ 5]^T) = [-1 \ 4]^T$ ,  $T([-4 \ 3]^T)$  can be computed if we can express  $[-4 \ 3]^T$  as a linear combination of vectors  $[1 \ -2]^T$  and  $[-2 \ 5]^T$ , i.e.; find  $a$  and  $b$  such that  $[-4 \ 3]^T = a[1 \ -2]^T + b[-2 \ 5]^T$ . We have:

$$\begin{aligned} \begin{cases} -4 &= a - 2b \\ 3 &= -2a + 5b \end{cases} &\implies \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \\ &\implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \end{bmatrix} \\ &= \frac{1}{5-4} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ -5 \end{bmatrix}, \end{aligned}$$

i.e.,  $a = -14$  and  $b = -5$ . It follows that

$$T([-4 \ 3]^T) = aT([1 \ -2]^T) + bT([-2 \ 5]^T) = -14[3 \ 4]^T - 5[-1 \ 4]^T = [-37 \ 76]^T.$$

7. This Markov chain with transition matrix  $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ , starting in state 1, has initial state vector  $S_0 = [1 \ 0]^T$ .
- (a) Since  $P^1 = P$  is the transition matrix with only nonzero entries, all powers of  $P$  also have nonzero entries. So, at least one power of  $P$  has nonzero entries, i.e., the chain is regular.
- (b)  $S_{m+1} = PS_m = P^{m+1}S_0$ , where  $S_0 = [1 \ 0]^T$ . So,  $S_1 = PS_0 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$  and the state-vector  $S_2$  after two transitions is  $S_2 = PS_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix}$ . So, the probability for the chain to be in state 1 after 2 transitions is  $\frac{5}{9}$ .

- (c) Since the chain is regular, it has a steady state-vector which is the only probability vector  $S$  solving the homogeneous system  $(I_2 - P)X = 0$ .

$$I_2 - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

( $r_1 \rightarrow 3 \times r_1/2$  and  $r_2 \rightarrow r_1 + r_2$ ). So, the homogeneous system  $(I_2 - P)X = 0$  has general solution  $X = t[1 \ 1]^T$ ,  $t \in \mathbb{R}$ . In particular the steady state-vector (which is a solution to the homo. syst.) has the form  $S = t[1 \ 1]$ , for some  $t \in \mathbb{R}$ . Since  $S$  is a probability vector, the sum of its entries is 1, therefore  $t = 1/2$  and  $S = \frac{1}{2}[1 \ 1]^T$ .