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Math 311

1. Let S be a set of vectors in $\Re^4$ such that $S = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

   a. Determine whether or not this set is linearly independent.

   b. Let U be the subspace spanned by S. Determine the dimension of U and construct and orthogonal basis for U.

   c. Extend the basis for U to an orthogonal basis for $\Re^4$.

2. Say whether or not the following subsets of M_{22} are subspaces of M_{22}:

   a. $U = \left\{ A \in M_{22}\text{: } AB = \bar{0},\ B \text{ is a fixed matrix in } M_{22} \right\}$.

   b. $U = \left\{ A \in M_{22}\text{: } A \text{ is not invertible in } M_{22} \right\}$

   c. $U = \left\{ A \in M_{22}\text{: } BAC = CAB \quad B \text{ and } C \text{ are fixed matrices in } M_{22} \right\}$

3. Let U be a non-empty subset of a vector space V. Show that U is a subspace of V if and only if $\boldsymbol{u_1} + \boldsymbol{a}\,\boldsymbol{u_2}$ lies in U for all $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ in U and all $\boldsymbol{a} \in \Re$.

4. If U and W are subspaces of a vector space V, let $Q = U \cup W = \{ \boldsymbol{v} \in V\text{: } \boldsymbol{v} \in U \text{ or } \boldsymbol{v} \in W \}$. Show that Q is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.

5. Let **A** be a square matrix with eigenvalue **v** and corresponding eigenvector **x**. Show that if **A** is invertible, then the eigenvalue of its inverse, **B**, is **1/v** and that **x** is the eigenvector of **B** associated with **1/v**.

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6. Suppose that λ is an eigenvalue of a square matrix A with non-zero eigenvector X, show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of the matrix $A^3 - 2A + 3I$.

7. Assume that the 2x2 matrix A is similar to an upper triangular matrix. If the $tr\ A = 0 = tr\ A^2$,n show that $A^2 = 0$.

8. Let V be a vector space and let $S = \{e_i\}_{i=1}^{n}$ be a subset of V.

   a. Define what is meant by the statement "S is a linearly independent set of vectors in the vector space V".

   b. Define what is meant by the statement "S is a basis for the vector space V".

   c. {**u, v, w, z**} is a linearly independent subset of a vector space V. Say whether or not the following sets are linearly independent or linearly dependent. In each case, justify your answer.
   (i) A = {**u-v, v-w, w-z, z-u**}

   (ii) B = {**u, u+v, u+v+w, u+v+w+z**}

9. Define what is meant by each of the following statements:
   a. V is a subspace of a vector space W.
   b. A vector space V has dimension equal to n.
   c. V and W are vector spaces. A linear transformation T from V to W is one-to-one.
   d. V and W are vector spaces. A linear transformation T from V to W is onto.
   e. Two vectors X and Y in $\mathfrak{R}^n$ are orthogonal.
   f. The matrix B has an eigenvalue λ and an eigenvector X.

10. Given that $S = \{e_i\}_{i=1}^{n}$ is an orthogonal set of n vectors in a vector space V, show that S is a linearly independent set of vectors.

11. Use the Gram Schmidt Orthogonalization lemma to transform the given set B into an

orthogonal basis for $\Re^4$.
$B = \{(1,1,1,0), (0,1,1,1), (1,0,1,1), (1,1,0,1)\}$.

12. Define what is meant by the statement " the nxn matrices A and B are similar".Show that if A and B are similar nxn matrices, then the following statements are true.

    a. $C_A(\lambda) = C_B(\lambda)$

    b. $\det A = \det B$

    c. $tr\ A = tr\ B$

13. Show that the vectors **X** and **Y** are orthogonal in $\Re^n$ if and only if $\|X + Y\| = \|X - Y\|$.

14. Given that T is a linear transformation from a vector space V to a vector space W, show that
    a. Ker T is a subspace of V.
    b. Im T is a subspace of W.
    c. T is one-to-one if and only if Ker T = {0}.

15. T is a mapping from $\Re^3$ to $\Re^4$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ x - y \\ x - z \\ 3y - 3z \end{pmatrix}$$

a. Show that T is a linear transformation.
b. Determine the transformation matrix associated with T.
c. Determine a basis for Ker T.
d. Determine a basis for Im T.
e. Determine the rank and nullity of T.

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16. The set B={(1,1,1), (1,0,1), (0,1,1)} and the set D={(1,0,2), (0,1,2), (1,2,0)} are ordered bases for the vector space $\Re^3$. Determine the transition matrix from B to D and the transition matrix from D to B.

17. a. Define what is meant by "The matrix P is orthogonal".
    b. Say whether or not the matrices given below are orthogonal or not. If the matrix is orthogonal find the inverse of the matrix.

$$i. \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad ii. \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -2\sqrt{6} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

18. Show that the matrix A is diagonalizable and find the matrix P which diagonalizes A.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$$

19. Find the bases for the row and column spaces of the matrix, A given by:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$$

Determine the rank of A.

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20. Consider the linear transformations:

$$T_1 : \Re^2 \to \Re^3 ; \qquad T_1 (x,y) = (x - y, x + 3y, 4y)$$

$$T_2 : \Re^3 \to \Re^2 ; \qquad T_2 (x,y,z) = (2x + y - 3z, x + 2y - z)$$

a. Show that T_1 is one to one but not onto.

b. Show that T_2 is onto but not one to one.

c. Find a formula for $T_2 \, T_1$.