

Department of Mathematics and Statistics
University of Calgary
Assignment

Math 311

1. Let $U, V,$ and W be vector spaces with $T: U \rightarrow V$ and $S: V \rightarrow W$ linear transformations. Prove that if ST is an isomorphism, then T is one-to-one and S is onto.

Definition: Let A and B be $n \times n$ matrices. A is similar to B ($A \sim B$) if and only if there exists an $n \times n$ invertible matrix P so that $P^{-1} A P = B$.

Question:

2. If $A, B,$ and C are $n \times n$ matrices. Show that
- If A is similar to B , then B is similar to A .
 - A is similar to A .
 - If A is similar to B and B is similar to C , then this implies that A is similar to C .

Definition: An $n \times n$ matrix A is diagonalizable exactly when there exists an invertible $n \times n$ matrix P such that $P^{-1} A P$ is a diagonal matrix.
[Notice that an equivalent definition might be: An $n \times n$ matrix A is diagonalizable exactly when A is similar to a diagonal matrix.]

Question:

3. Give an example of
- a 2×2 diagonalizable matrix.
 - a 3×3 diagonalizable matrix.
 - a 2×2 matrix which is not diagonalizable..

Definition: If A is an $n \times n$ matrix, a number λ is an eigenvalue of A if $A X = \lambda X$ for some non-zero column $X \in \mathbb{R}^n$.

Example: Consider the 2×2 matrix $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$.

If $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $A X = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Notice that $\lambda = -1$ is an eigenvalue for the matrix A and $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A which corresponds to the eigenvalue -1 .

Department of Mathematics and Statistics
University of Calgary
Assignment

Math 311

Remark: Given an $n \times n$ matrix A , to find the eigenvalues, λ for A , we must have
 $AX = \lambda X$ for some $\mathbf{0} \neq X \in \mathbb{R}^n$.

This means that

$$\begin{aligned} AX - \lambda X &= \mathbf{0} \\ \therefore AX - \lambda I_n X &= \mathbf{0} \\ \therefore (A - \lambda I_n) X &= \mathbf{0} \end{aligned} \tag{1}$$

Therefore we need to find λ so that the system of equations in (1) will have non-trivial solutions. This means that the matrix $A - \lambda I_n$ must not be invertible. Consequently we must have $\det(A - \lambda I_n) = 0$.

$\det(A - \lambda I_n)$ is the characteristic polynomial of the matrix A .

Example: If $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$ and λ is an eigenvalue of A , then,

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix} \\ \therefore A - \lambda I_n &= \det \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix} \\ &= (1 - \lambda)(2 - \lambda) - 6 \\ &= \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4. \end{aligned}$$

Solving the equation $\det(A - \lambda I_n) = 0$, we get $\lambda = 4$ or $\lambda = -1$.

We conclude that the matrix A has two eigenvalues -1 and 4 .

Department of Mathematics and Statistics
University of Calgary
Assignment

Math 311

Questions:

4. In each case determine the eigenvalues of the matrices:

$$(i) \quad A = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ -1 & 1 & 4 \end{pmatrix} \qquad (ii) \quad B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$
$$(iii) \quad C = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \qquad (iv) \quad D = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ +4 & 4 & 0 \end{pmatrix}$$

Remark: Let A be an $n \times n$ matrix with an eigenvalue λ . Let $E_\lambda = \{X \in \mathbb{R}^n : AX = \lambda X\}$. Then every $X \in E_\lambda$, $X \neq \mathbf{0}$ is an eigenvector associated with the eigenvalue λ . [E_λ is called an eigenspace.]

Question:

5. Prove that E_λ as defined above is a subspace of \mathbb{R}^n .
6. Using the matrices given in question 3, determine the eigenvectors associated with each eigenvalue and describe the resulting eigenspaces. In each instance determine the dimension of the eigenspace.
7. If A and B are $n \times n$ matrices and if A is similar to B , show that
 - a. $\det A = \det B$.
 - b. $\text{tr } A = \text{tr } B$.
 - c. A and B have the same characteristic polynomial.
 - d. A and B have the same eigenvalues.
 - e. If A is diagonalizable, then B is diagonalizable.

Department of Mathematics and Statistics
University of Calgary
Sheet 6

Math 311

1. Let \mathbf{x} , \mathbf{y} , \mathbf{z} be vectors in \mathbb{R}^n . Show that
 - a. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
 - b. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
 - c. $(k\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (k\mathbf{y}) = k(\mathbf{x} \cdot \mathbf{y})$
2. Given that \mathbf{x} , $\mathbf{y} \in \mathbb{R}^n$, show that $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$ if and only if $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$.
3. Given that \mathbf{x} , $\mathbf{y} \in \mathbb{R}^n$, show that $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = 0$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.
4. Show that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$.
5. Let \mathbf{x} , \mathbf{y} be vectors in \mathbb{R}^n , show that
 - a. $\mathbf{x} \cdot \mathbf{y} = \frac{1}{4} (\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2)$
 - b. $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \frac{1}{2} (\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2)$
6. Find a and b so that the vectors \mathbf{x} , \mathbf{y} , $\mathbf{z} \in \mathbb{R}^3$ are orthogonal.
$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix}$$
7. The set $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an orthogonal basis for \mathbb{R}^3 . Express
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$
 as a linear combination of the vectors in B . Determine an orthonormal basis for \mathbb{R}^3 .

Department of Mathematics and Statistics
University of Calgary
Sheet 6

Math 311

8. Given the set $B = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ in \mathbb{R}^4 , determine a, b, c, and d so that

the set B is an orthogonal set.

9. In each case either show that the statement is true or give an example showing that the statement is false.
- A set in \mathbb{R}^4 is linearly independent if and only if the set is an orthogonal set.
 - If $\{ \mathbf{x}, \mathbf{y} \}$ is an orthogonal set in \mathbb{R}^n , then $\{ \mathbf{x}, \mathbf{x} + \mathbf{y} \}$ is also an orthogonal set.
 - If $\{ \mathbf{x}_i \}_{i=1}^2$ and $\{ \mathbf{y}_j \}_{j=1}^3$ are both orthogonal sets in \mathbb{R}^n , and if $\mathbf{x}_i \cdot \mathbf{y}_j = 0$ for all i and j , then $\{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \}$ is an orthogonal set in \mathbb{R}^n .