Numbers

Classroom notes for PMAT 435, Fall 2005

Abstract

This is a short outline of how to construct the rational numbers and the real numbers starting with the integers.

1 The integers

We assume that the set of integers, $\mathbb{I}$, are given; that they are closed under addition, $+$, and multiplacation, $\cdot$ and the unary operation $-$; that $<$ is a binary relation on $\mathbb{I}$, that the operations of $+$ and $\cdot$ and $-$, the relation $<$ and the integers 0 and 1 satisfy the following algebraic and order properties for all $a, b, c \in \mathbb{I}$:

P1: $a + b = b + a$.

P2: $(a + b) + c = a + (b + c)$.

P3: $0 + a = a$.

P4: $a + (-a) = 0$.

P5: $a \cdot b = b \cdot a$.

P6: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

P7: $a \cdot 1 = a$.

P8: If $a \cdot c = b \cdot c$ and $c \neq 0$ then $a = b$.

P9: $a \cdot (b + c) = a \cdot b + a \cdot c$.

P10: $a = b$ or $a < b$ or $a > b$.

P11: $a < b < c$ implies $a < c$.

P12: $a < b$ implies $a + c < b + c$.

P13: $a < b$ and $c > 0$ implies $a \cdot c < b \cdot c$.

P14: $0 < 1$.

P15: $0 < n$ if and only if $-n < 0$.

P16: If $0 \leq a \leq 1$ then $a = 0$ or $a = 1$.

P17: Let $\mathbb{N} := \{n \in \mathbb{I} : 0 < n\}$. Then $\emptyset \neq S \subseteq \mathbb{N}$ implies that S has a minimum.

Define: $a - b := a + (-b)$.

We obtain $a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a$. Adding $-a$ we get $(-a) + (a + a \cdot 0) = (-a) + a$ and hence $a \cdot 0 = 0$.

Also $a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a = (1 + (-1)) \cdot a = 0 \cdot a = 0$ and adding $(-a)$ we get $(-1) \cdot a = -a$.

Also $-(-a) + (-a) = 0$. Adding a we get $-(-a) = a$, which with the previous result implies that $(-1) \cdot (-1) = 1$.

Let $a \cdot b = 0$ and assume $b \neq 0$. Then $a \cdot b = 0 \cdot b$ which implies according to property P8 that $a = 0$.

If $a < b$ and $a_1 < b_1$ then $a + a_1 < b + a_1 < b + b_1$ according to P1 and P12. Hence we obtain from P14 that the numbers $1, 1 + 1, 1 + 1 + 1, \ldots$, usually written $1, 2, 3, \ldots$ are positive and hence elements of $\mathbb{N}$. It follows, but we will not supply the proof here, that given a natural number $b > 1$ every natural number can be written in base b. We will as usual write the natural numbers in base 10.

Note that P15 is equivalent to the following induction principle: If $S \subseteq \mathbb{N}$ and $1 \in S$ and $n \in S$ implies $n + 1 \in S$ then $S = \mathbb{N}$.

The integer a *divides* the integer b, or is a *factor of* b, if there exists an integer q so that $aq = b$. The natural number p is a *prime number* if the only natural numbers which are factors of p are 1 and p itself. The numbers a and b are *relatively prime* if the only common factors of a and b are the numbers 1 and -1. (Note that 0 and 1 and 0 and -1 are relatively prime but 0 and 5 are not relatively prime.)

Without providing the proof here we will use the following facts from number theory:

Theorem 1.1. *1. If the prime number p is a factor of the product ab of two integers a and b then p is a factor of a or a factor of b.*

2. Every natural number can be factorized into primes and this factorization is unique. (Prime Factorization Theorem.)

3. *Given two integers a and $b \neq 0$ there exists a unique integer d and unique integers a_1 and $b_1 > 0$ with $a_1 d = a$ and $b_1 d = b$ so that a_1 and b_1 are relatively prime. The natural number $|d|$ is the* greatest common divisor *of a and b.*

2 The rational numbers

The set of *fractions*, $\mathbb{F}$, is the set:

$$\mathbb{F} := \{\frac{a}{b} : a \in \mathbb{I} \text{ and } b \in \mathbb{I} \text{ and } b \neq 0\}.$$

We define the binary relation $\sim$ on $\mathbb{F}$ as follows:

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{if} \quad a \cdot d = b \cdot c.$$

The relation $\sim$ is an equivalence relation on $\mathbb{F}$ because:

1. $\frac{a}{b} \sim \frac{a}{b}$ because $a \cdot b = a \cdot b$.
2. If $\frac{a}{b} \sim \frac{c}{d}$ then $\frac{c}{d} \sim \frac{a}{b}$ because if $a \cdot d = b \cdot c$ then $c \cdot b = d \cdot a$.
3. If $\frac{a}{b} \sim \frac{c}{d} \sim \frac{e}{f}$ then $a \cdot d = b \cdot c$ and $c \cdot f = e \cdot d$. It follows that $a \cdot d \cdot f = b \cdot c \cdot f = b \cdot e \cdot d$ and hence $a \cdot f = b \cdot e$ which implies $\frac{a}{b} \sim \frac{e}{f}$.

The set of rational numbers $\mathbb{Q}$ is the set of $\sim$-equivalence classes of $\mathbb{F}$. Given a fraction $\frac{a}{b}$ we denote the $\sim$-equivalence class containing $\frac{a}{b}$ by $[\frac{a}{b}]$. Note that $\frac{a'}{b'} \in [\frac{a}{b}]$ if and only if $\frac{a'}{b'} \sim \frac{a}{b}$. Note also that $\frac{0}{b} \sim \frac{a_1}{b_1}$ if and only if $a_1 = 0$.

Lemma 2.1. *Every equivalence class $[\frac{a}{b}]$ of fractions with a contains a unique fraction $\frac{a_1}{b_1}$ so that a_1 and b_1 are relatively prime and b_1 is positive. There exists a natural number d so that $a = da_1$ and $b = db_1$.*

Proof. It follows from Theorem 1.1 that there are unique relative prime integers a_1 and $b_1 > 0$ and an integer d so that $a_1 d = a$ and $b_1 d = b$ and hence $\frac{a}{b} \sim \frac{a_1}{b_1}$. □

Definition 2.1. *The fraction $\frac{a}{b}$ is* reduced *if $b > 0$ and the numbers a and b are relatively prime.*

Corollary 2.1. *Every equivalence class of fractions contains exactly one reduced fraction. If $\frac{a_1}{b_1}$ is a reduced fraction and $\frac{a_1}{b_1} \sim \frac{a}{b}$ then there exists an integer d with $da_1 = a$ and $db_1 = b$.*

Lemma 2.2. *Let $k \neq 0$ be an integer then $\frac{a}{b} \sim \frac{ka}{kb}$ for all fractions $\frac{a}{b}$. If $\frac{a}{b} \sim \frac{c}{d}$ then $\frac{la}{b} \sim \frac{lc}{d}$ for all integers l. Also, $\frac{0}{b} \sim \frac{a}{d}$ if and only if $0 = a$.*

Proof. It is clearly the case that $akb = bka$.

The equivalence $\frac{a}{b} \sim \frac{c}{d}$ implies $ad = bc$ implies $lad = blc$ implies $\frac{la}{b} \sim \frac{lc}{d}$.

We have $\frac{0}{b} \sim \frac{0}{d}$ because $0d = 0b$. If $\frac{0}{b} \sim \frac{a}{d}$ then $0 = 0d = ab$, which implies $a = 0$ because $b \neq 0$. □

Lemma 2.3.

$$\frac{a}{b} \sim \frac{a_1}{b_1} \text{ and } \frac{c}{d} \sim \frac{c_1}{d_1} \text{ implies } \frac{ad+cb}{bd} \sim \frac{a_1d_1 + c_1b_1}{b_1d_1}.$$

Proof. Note that $ab_1 = a_1b$ and $cd_1 = c_1d$. Hence

$$(ad + cb)b_1d_1 = adb_1d_1 + cbb_1d_1 = bda_1d_1 + bdc_1b_1 = bd(a_1d_1 + c_1b_1).$$

□

Definition 2.2. *Let $[\frac{a}{b}]$ and $[\frac{c}{d}]$ be two rational numbers. Then*

$$\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] := \left[\frac{ac+bd}{bd}\right].$$

Lemma 2.4. *The addition, $+$, of the rationals satisfies properties P1, P2, P3 and P4 with $[\frac{0}{1}]$ the zero element and $-[\frac{a}{b}] := [\frac{-a}{b}]$ the additive inverse.*

Proof. Exercise. □

Lemma 2.5.

$$\frac{a}{b} \sim \frac{a_1}{b_1} \text{ and } \frac{c}{d} \sim \frac{c_1}{d_1} \text{ implies } \frac{ac}{bd} \sim \frac{a_1c_1}{b_1d_1}.$$

Proof. Exercise.

□

Definition 2.3. *Let $[\frac{a}{b}]$ and $[\frac{c}{d}]$ be two rational numbers. Then*

$$\left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right] := \left[\frac{ac}{bd}\right].$$

Lemma 2.6. *Let $a \neq 0$ and $\frac{a}{b} \sim \frac{a_1}{b_1}$. Then $a_1 \neq 0$ and $\frac{b}{a} \sim \frac{b_1}{a_1}$.*

Proof. Exercise. □

Definition 2.4. *Let $[\frac{a}{b}]$ with $a \neq 0$ be a rational number. Then $[\frac{a}{b}]^{-1} := [\frac{b}{a}]$.*

Let P'8 be the property:

P'8: $r \cdot r^{-1} = 1$.

Lemma 2.7. *The product, $\cdot$ and sum, $+$, of the rationals satisfies properties P5, P6, P7, P'8 and P9 with $[\frac{1}{1}]$ the one element.*

Proof. Exercise. □

Lemma 2.8. *Let $\frac{a}{b} \sim \frac{a_1}{b_1}$ and $\frac{c}{d} \sim \frac{c_1}{d_1}$ with b, b_1, d and d_1 all positive. Then $ad < cb$ if and only if $a_1 d_1 < c_1 b_1$.*

Proof.

$$ad < cb \text{ iff } ab_1dd_1 < cbb_1d_1 \text{ iff } a_1bdd_1 < c_1bb_1d \text{ iff } a_1d_1 < c_1b_1.$$

□

Definition 2.5. *Let $b > 0$ and $c > 0$. Then $\frac{a}{b} < \frac{c}{d}$ if $ad < bc$.*

Lemma 2.9. *The operations plus, $+$, and times, $\cdot$, and the relation less than, $<$, of the rationals satisfy properties P10, P11, P12, P13, P14 and P15.*

Proof. Exercise. □

Lemma 2.10. *The function $f : \mathbb{I} \to \mathbb{Q}$ given by $f(n) := [\frac{n}{1}]$ is one to one and has for all $a, b \in \mathbb{I}$ the properties:*

$$f(a + b) = f(a) + f(b) \text{ and } f(a \cdot b) = f(a) \cdot f(b) \text{ and} \tag{1}$$

$$a < b \text{ iff } f(a) < f(b). \tag{2}$$

Proof. Exercise. □

Note that the operations $+$ and $\cdot$ in (1) and the relation $<$ in (2) on the left of the $=$ sign are operations a relation between integers and on the right hand side of the $=$ sign are the operations and the relation defined above between rationals.

We will adopt the usual agreement to write $\frac{a}{b}$ instead of $[\frac{a}{b}]$. It follows from Lemma 2.10 that we can also adopt the convention to write a for $\frac{a}{1}$. We also adopt the convention to write $\frac{a}{b} = \frac{c}{d}$ to mean that $[\frac{a}{b}] = [\frac{c}{d}]$.

Lemma 2.11. *For every rational number r there exists an integer n with $n > r$.*

Proof. We may assume without loss that $r = \frac{a}{b}$ with $b > 0$. Then $\frac{a}{b} < a+1$.
□

3 The real numbers

Definition 3.1. *The sequence $\{a_n\}$ of rational numbers is a* rational Cauchy sequence *if for every rational $\epsilon > 0$ there exists an index n^* so that for all $m, n > n^*$ we have:*

$$|a_n - a_m| < \epsilon.$$

We denote by $\mathbb{CA}$ be the set of rational Cauchy sequences.

Lemma 3.1. *If the sequence $\{a_n\}$ of rationals is monotone increasing and bounded above then it is a rational Cauchy sequence.*

Proof. If not, then there exists an $\epsilon > 0$ so that for every index n^* there are two indices n and m with $n, m > n^*$ so that $|a_n - a_m| \geq \epsilon$. This implies that for every index n^* there are two indices n and m so that $n < n < m$ and $a_n^* \leq a_n < a_n + \epsilon \leq a_m$. Hence there exists for every index n and index m with $n < m$ and $a_n + \epsilon \leq a_m$. But this contradicts the assumption that the sequence $\{a_n\}$ is bounded above.

□

Definition 3.2. *Let $\{a_n\}$ and $\{b_n\}$ be two rational Cauchy sequences in $\mathbb{CA}$. Then $\{a_n\} \sim \{b_n\}$ if for every rational $\epsilon > 0$ there exists an index n^* so that for all $n > n^*$ we have:*

$$|a_n - b_n| < \epsilon.$$

Lemma 3.2. *The relation $\sim$ is an equivalence relation on $\mathbb{CA}$.*

Proof. It follows easily from the definition that $\{a_n\} \sim \{a_n\}$ and that $\{a_n\} \sim \{b_n\}$ implies $\{b_n\} \sim \{a_n\}$.

Let $\{a_n\} \sim \{b_n\} \sim \{c_n\}$ and let ϵ be given. Let the index n_1 be such $|a_n - b_n| < \frac{\epsilon}{2}$ for all $n > n_1$ and let n_2 be such that $|b_n - c_n| < \frac{\epsilon}{2}$ for all $n > n_2$. Let $n^* := \max\{n_1, n_2\}$. Then:

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $n > n^*$. □

Definition 3.3. *Let $\{a_n\} \in \mathbb{CA}$. We denote by $[\{a_n\}]$ the $\sim$-equivalence class containing the sequence $\{a_n\}$. The set $\mathbb{R}$ of* real numbers *is the set of $\sim$-equivalence classes of $\mathbb{CA}$.*

Lemma 3.3. *For every sequence $\{a_n\} \in \mathbb{CA}$ there exists a positive rational number A so that $-A < a_n < A$ for all indices n.*

Proof. Exercise. Similar to the proof that every Cauchy sequence is bounded, given in the textbook. □

This number A is a *positive bound of the sequence* $\{a_n\}$.

Lemma 3.4. *Let $\{a_n\} \in \mathbb{CA}$ a sequence with $\{a_n\} \not\sim \{0\}$ the constant 0-sequence. Then there exists a positive rational number L and an index n^* so that $L < |a_n|$ for all $n > n^*$.*

Proof. Assume for a contradiction that for every $\epsilon > 0$ there are infinitely many indices n so that $|a_n| < \epsilon$. Then, given $\epsilon > 0$.

There are infinitely many indices n so that $|a_n| < \frac{\epsilon}{2}$. There exists an n^* so that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m > n^*$. There is an $n_1 > n^*$ with $|a_{n_1}| < \frac{\epsilon}{2}$. Hence:

$$|a_m - 0| = |a_m| = |a_m - a_{n_1} + a_{n_1}| \leq |a_m - a_{n_1}| + |a_{n_1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{a_n\} \sim \{0\}$ in contradiction to our assumption. □

Definition 3.4. *Let a and b be rational numbers. Then $a//b$ is equal to a if $b = 0$ and equal to $\frac{a}{b}$ otherwise.*

Lemma 3.5. *Let $\{an\}, \{b_n\} \in \mathbb{CA}$ and let k be a rational number. Then the sequences $\{ka_n\}$ and $\{a_n + b_n\}$ and $\{a_n b_n\}$ are elements of $\mathbb{CA}$. If $a_n \neq 0$ for all indices n and $\{a_n\} \not\sim \{0\}$, then the sequences $\{1//a_n\}$ and $\{b_n//a_n\}$ are elements of $\mathbb{CA}$.*

Proof. Let $\epsilon > 0$ be given and let $k' \geq k$ be a positive rational. There exists an index n^* so that $|a_n - a_m| < \frac{1}{k'}\epsilon$ for all $n, m > n^*$. Then $|ka_n - ka_m| = |k||a_n - a_m| = k'|a_n - a_m| < k'\frac{1}{k'}\epsilon = \epsilon$.

The case $\{a_n + b_m\}$ is left as an exercise.

Let A be a positive bound of the sequence $\{a_n\}$ and let B be a positive bound of the sequence $\{b_n\}$. Given $\epsilon > 0$ there is an index n_1 so that $|b_n - b_m| < \frac{\epsilon}{2A}$ for all $n, m > n_1$. There is an index n_2 so that $|a_n - a_m| < \frac{\epsilon}{2B}$ for all $n, m > n_2$. Let $n^* := \max\{n_1, n_2\}$. Then:

$$\begin{aligned} &|a_n b_n - a_m b_m| = |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \leq \\ &|a_n||b_n - b_m| + |b_m||a_n - a_m| \leq A|b_n - b_m| + B|a_n - a_m| < \\ &A\frac{\epsilon}{2A} + B\frac{\epsilon}{2B} = \epsilon, \end{aligned}$$

for all $n > n^*$.

Let n_1 and $L > 0$ be such that $|a_n| > L$ for all $n > n_1$. Note that if $n > n_1$ then $1//a_n = \frac{1}{a_n}$. Let $\epsilon > 0$ be given. There exists an index $n^* > n_1$ so that $|a_n - a_m| < L^2\epsilon$. Then:

$$\left|\frac{1}{a_n} - \frac{1}{a_m}\right| = \left|\frac{a_m - a_n}{a_n a_m}\right| = \frac{1}{|a_n||a_m|}|a_n - a_m| < \frac{1}{L^2}L^2\epsilon = \epsilon,$$

for all $n > n^*$.

Combining the last two results above we conclude that the sequence $\{b_n//a_n\}$ is an element of $\mathbb{CA}$. □

Lemma 3.6. *Let $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$. Then $\{a_n + b_n\} \sim \{a'_n + b'_n\}$.*

Proof. Exercise.

□

Definition 3.5. *Let $\{a_n\}, \{b_n\} \in \mathbb{CA}$. Then $[\{a_n\}] + [\{b_n\}] := [\{a_n + b_n\}]$.*

Lemma 3.7. *The addition, $+$, of the reals satisfies properties P1, P2, P3 and P4, with the sequence $\{0\}$ as the zero element and $-[\{a_n\}] = [\{-a_n\}]$ as the additive inverse.*

Proof. Exercise. □

Lemma 3.8. *Let $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$. Then $\{a_n b_n\} \sim \{a'_n b'_n\}$.*

Proof. Exercise.

□

Definition 3.6. *Let $\{a_n\}, \{b_n\} \in \mathbb{CA}$. Then $[\{a_n\}] \cdot [\{b_n\}] := [\{a_n b_n\}]$.*

Lemma 3.9. *The product, $\cdot$, of the reals satisfies properties P5, P6, P7 and P'8, with the sequence $\{1\}$ as the one element and if $[\{a_n\}] \neq [\{0\}]$ with $[\{a_n\}]^{-1} = [\{1//a_n\}]$ as the multiplicative inverse.*

Proof. Exercise. □

Lemma 3.10. *Let $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$. If there exists a rational number $d > 0$ and index n_1 so that $a_n + d \leq b_n$ for all $n > n_1$ then there exists an index n_2 and a rational number $d' > 0$ so that $a'_n + d' \leq b'_n$ for all $n > n_2$.*

Proof. Exercise. □

Definition 3.7. *We write* $[\{a_n\}] < [\{b_n\}]$ *if there exists an index* n^* *and a rational number* $d > 0$ *so that* $a_n + d \leq b_n$ *for all* $n > n^*$.

Lemma 3.11. *The operations plus, +, and times, ·, of reals and the relation less than, <, of reals satisfy the properties P10, P11, P12 and P13.*

Proof. Exercise. □

Lemma 3.12. *Let* $f : \mathbb{Q} \to \mathbb{R}$ *be the function which associates with every rational number* r *the constant sequence* $\{r\}$. *The function* f *is one-to-one and has for all* $a, b \in \mathbb{Q}$ *the properties:*

$$f(a+b) = f(a) + f(b) \text{ and } f(a \cdot b) = f(a) \cdot f(b) \text{ and} \tag{3}$$
$$a < b \text{ iff } f(a) < f(b). \tag{4}$$

If $b \neq 0$ *then* $f(b^{-1}) = (f(b))^{-1}$.

Proof. Exercise. □

Lemma 3.12 alowes us to adopt the usual notation for calculations with reals, in particular to write r instead of $[\{r\}]$ for the rational r. Note also that if r and s are two reals then $r - s := r + (-s)$. Except for the completeness axiom we have established all of the axioms of the reals indicated in the textbook. Hence we can use the established definitions and consequences of those axioms which do not use the completeness axiom.

Lemma 3.13. *For every real number* a *there exists a rational number* A *with* $a < A$.

Proof. Exercise. □

A set together with a binary relation $<$ which has properties P10 and P11 is a *total order*. It follows that the set of reals and every subset of the set of reals are a total order. We will take it given that every finite total order has a maximum. (It comes as a surprise to most students that the notions of finite and infinite need a careful definition in Set Theory. We assume that those notions are well defined and have the usual generally accepted properties.)

Let S be a non empty set of reals. The interval $[x, y]$ *restricts* the set S if:

1. x and y are rational numbers.
2. There are infinitely many elements $s \in S$ with $x < s$.
3. There is no element $s \in S$ with $y \leq s$.

4. The elements of S which are in $[x, y]$ do not have a maximum.

Note that if $[x, y]$ is a restricting interval of S then either $[x, \frac{y+x}{2}]$ or $[\frac{y+x}{2}, y]$ is a restricting interval.

Theorem 3.1 (Completeness Theorem)**.** *Every non empty subset S of the reals which is bounded above has a supremum.*

Proof. If S has a maximum then this maximum is the supremum and we are done. Assume that S does not have a maximum.

Let the rational $M > 0$ be an upper bound of S. Let t be an element of S and $A = \max\{|t|, M\}$. Then A is an upper bound of S and the interval $[-A, A]$ contains the element t of S and hence infinitely many elements of S. (Otherwise S would have a maximum.) It follows that the interval $[-A, A]$ is a restricting interval of S.

Hence there exist sequences $\{a_n\}$ and $\{b_n\}$ of rationals so that the sequence $\{a_n\}$ is monotonically increasing, the sequence $\{b_n\}$ is monotonically decreasing, all of the intervals $[a_n, b_n]$ are restricting intervals of S, the sequence $\{a_n\}$ is bounded and hence according to Lemma 3.1 in $\mathbb{CA}$. Note that $b_n - a_n \leq \frac{2A}{2^n}$.

We claim that the sequence $\{a_n\}$ is the supremum of the set S. We prove first that $[\{a_n\}]$ is an upper bound of S. If not then there exists an element $[\{c_n\}] = s \in S$ so that $[\{a_n\}] < [\{c_n\}]$. This implies that there exists a rational $d > 0$ and an index n_1 so that $a_n + d \leq c_n$ for all $n > n_1$. There exists an index n_2 so that $\frac{2A}{2^n} < \frac{d}{2}$. (Prove this.) If $n > \max\{n_1, n_2\}$ then because $b_n - a_n \leq \frac{2A}{2^n}$ we get $b_n + \frac{d}{2} \leq a_n + \frac{2A}{2^n} + \frac{d}{2} < a_n + d \leq c_n$. It follows that $b_n < [\{c_n\}] = s$ in contradiction that the interval $[a_n, b_n]$ is a restrictive interval of S.

We prove next that $\{a_n\}$ is the smallest upper bound of S. Assume for a contradiction that there exists a real $[\{c_n\}] < [\{a_n\}]$ which is an upper bound of S. This implies that there is an index n_1 and a rational $d > 0$ so that $c_n + d \leq a_n$ for all $n > n_1$. For every element $s = [\{x_n\}] \in S$ there exists an index n_s so that for all $n > \max\{n_x, n_1\}$ we have $x_n < c_n + \frac{d}{2}$. (Prove this.) Hence $x_n + \frac{d}{2} < c_n + d \leq a_n$, which in turn implies that $s < [\{a_n\}]$ for all $s \in S$ in contradiction to the fact that $[a_n, b_n]$ is a restricting sequence of S.

□

4 The functions $\exp(x)$ and a^x

Let $x \in \mathbb{R}$ and

$$s_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{5}$$

Theorem 4.1. *The sequence $s_n(x)$ converges for every real number x.*

Proof. We will show that the sequence $s_n(x)$ is a Cauchy sequence. Let $r \in \mathbb{N}$ with $r > |x|$ and let $\epsilon > 0$ be given. The sequence

$$\{\left(\frac{|x|}{r}\right)^n\}$$

tends to 0 with n to ∞ and hence there is a number $h \in \mathbb{N}$ with:

$$\left(\frac{|x|}{r}\right)^h < \frac{(r)!}{|x|^r}\left(1 - \frac{|x|}{r}\right)\epsilon \text{ and hence with } \frac{|x|^r}{r!}\left(\frac{|x|}{r}\right)^h \frac{1}{1-\frac{|x|}{r}} < \epsilon. \tag{6}$$

Let $n^* > r + h$ then:

$$|s_m(x) - s_n(x)| \le \sum_{r+h}^{\infty} \frac{|x|^k}{k!} =$$

$$\sum_{k=0}^{\infty} \frac{|x|^{r+h+k}}{(r+h+k)!} \le \frac{|x|^r}{r!}\left(\frac{|x|}{r}\right)^h \sum_{k=0}^{\infty}\left(\frac{|x|}{r}\right)^k = \frac{|x|^r}{r!}\left(\frac{|x|}{r}\right)^h \frac{1}{1-\frac{|x|}{r}} < \epsilon.$$

□

Theorem 4.2. *Let $0 \le x, y \in \mathbb{R}$. Then*

$$\exp(x) \cdot \exp(y) = \exp(x+y).$$

Proof.

$$\exp(x) \cdot \exp(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!} =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^k}{n!} = \exp(x+y).$$

□

Definition 4.1. $e^x := \exp(x)$

Theorem 4.3. *Let $x, y \in \mathbb{R}$. Then:*

1. $e^x e^y = e^{x+y}$.
2. $e^{-x} = \frac{1}{e^x}$.
3. $e^1 = a$.
4. $e^0 = 1$.

Proof. Exercise.

□

Lemma 4.1.

$$\lim_{x \to 0} \exp(x) = 1.$$

Proof. Let $0 \leq x < 1$. Then

$$1 \leq \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \leq 1 + x + \sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k = 1 + x + \frac{\frac{x}{2}}{1 - \frac{x}{2}}.$$

Hence $\lim_{x \to 0^+} \exp(x) = 1$. Also $\lim_{x \to 0^-} \exp(x) = 1$ because $\exp(-x) = \frac{1}{\exp(x)}$. Hence $\lim_{x \to 0} \exp(x) = 1$.

□

Lemma 4.2. *The function* $\exp(x)$ *is continuous.*

Proof. Because $\exp(x)$ is defined on all of $\mathbb{R}$ it suffices to show that $\lim_{x \to a} \exp(x) = \exp(a)$ for all $a \in \mathbb{R}$. That is, we have to prove that $\lim_{h \to 0} \exp(x + h) = \exp(x)$ for all $x \in \mathbb{R}$.

$$\lim_{h \to 0} \exp(x + h) = \lim_{h \to 0} \exp(x) \exp(h) = \exp(x) \lim_{h \to 0} \exp(h) = \exp(x).$$

□

Lemma 4.3. $\lim_{x \to \infty} \exp(x) = \infty$ *and* $\lim_{x \to -\infty} \exp(x) = 0$.

Proof. If $x > 0$ then $1 + x < \exp(x)$. □

Lemma 4.4. *If* $x < y$ *then* $\exp(x) < \exp(y)$.

Proof. Exercise. □

Theorem 4.4. *The function $e^x = \exp(x)$ is a monotonically increasing continuous function which maps the set of real numbers onto the set of positive real numbers.*

Proof. Follows from Lemma 4.2 and Lemma 4.3 and Lemma 4.4.

□

Definition 4.2. *Let $0 < a \in \mathbb{R}$. Then $\ln a$ is the unique number x so that $e^x = a$*

Note that $e^{\ln a} = a = \ln e^a$.

Definition 4.3. *Let $0 < a \in \mathbb{R}$ and $x \in \mathbb{R}$. Then $a^x := e^{x \ln a}$.*

Lemma 4.5. $(e^x)^y = e^{xy}$.

Proof.

$$(e^x)^y = e^{y \ln e^x} = e^{xy}.$$

□

Theorem 4.5. *Let $0 < a \in \mathbb{R}$ and $x, y \in \mathbb{R}$. Then:*

1. $a^x a^y = a^{x+y}$.
2. $a^{-x} = \frac{1}{a^x}$.
3. $a^1 = a$.
4. $a^0 = 1$.
5. $(a^x)^y = a^{xy}$.

Proof. Items 1 to 4 are easy to see. We will prove Item 5.

$$(a^x)^y = e^{y \ln a^x} = e^{y \ln e^{x \ln a}} = e^{yx \ln a} = a^{xy}.$$

□

Note that $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{n}{n}} = a^1 = a$ and hence that $a^{\frac{1}{n}}$ is indeed the n^{th} root of a.

Theorem 4.6. *Let $x, y \in \mathbb{R}$. Then:*

1. $ln 1 = 0$.
2. $\ln(xy) = \ln x + \ln y$.

3. $\ln x^y = y \ln x$.

Proof. Item 2 and 3 follow from:

$$e^{\ln(ab)} = ab = e^{\ln a}e^{\ln b} = e^{\ln a + \ln b} \quad \text{and} \quad e^{\ln x^y} = x^y = \left(e^{\ln x}\right)^y = e^{y \ln x}$$

and the monotonicity of e^x. □

Lemma 4.6.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Proof. For every $n \in \mathbb{N}$ there exists a real number h_n so that $1 + h_n = n^{\frac{1}{n}}$. Then $(1 + h_n)^n = n$. It follows that

$$\sum_{i=0}^{n} \binom{n}{i} h_n^i = n$$

and hence that

$$\frac{n(n-1)}{2} h_n^2 \leq n.$$

This implies

$$h_n \leq \sqrt{\frac{2n}{n(n-1)}}$$

and hence that $\lim_{n\to\infty} h_n = 0$ according to the sandwhich theorem. Then:

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (1 + h_n) = 1.$$

□

Let $a \geq 1$ then eventually $n > a$ and $1 \leq \sqrt[n]{a} \leq \sqrt[n]{n}$ and hence $\lim_{n\to\infty} \sqrt[n]{a} = 1$ according to the sandwhich theorem. If $0 < a < 1$ then $\frac{1}{a} > 1$.

Theorem 4.7. *Let $x \in \mathbb{R}$, then:*

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Proof. Let $t_n(x) := \left(1 + \frac{x}{n}\right)^n$. For $2 \leq n, k \in \mathbb{N}$ and $k \leq n$ let

$$f(n,k) := \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{3}{n}\right) \cdot \dots \cdot \left(1 - \frac{k-1}{n}\right)$$

and $f(n,0) = f(n,1) = 1$. Then $f(n,k) \leq 1$ and $\lim_{n\to\infty} f(n,k) = 0$ and

$$\binom{n}{k} \cdot \frac{1}{n^k} = \frac{1}{k!} \cdot f(n,k)$$

and

$$t_n(x) = \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} = \sum_{k=0}^{n} \frac{x^k}{k!} \cdot f(n,k). \tag{7}$$

It follows from Equations 5 and 7 that $t_n(x) \leq e^x$. For $2 \leq m < n$ let

$$s'_m(x,n) := \sum_{k=0}^{m} \frac{x^k}{k!} \cdot f(n,k).$$

Note that $s'_m(x,n) \leq s_m(x)$ and that $\lim_{n\to\infty} s'_m(x,n) = s_m(x)$ and that $t_n(x) \leq s'_m(x,n)$.

In order to prove that $\lim_{n\to\infty} = e^x$ let $\epsilon > 0$ be given. Let m^* be such that $e^x - s_m(x) < \frac{\epsilon}{2}$ for all $m > m^*$. Fix $m > m^*$ and n_1 so that $s_m(x) - s'_m(x,n) < \frac{\epsilon}{2}$ for all $n > n_1$. Let $n^* := \max\{m, n_1\}$. For $n > n^*$ we have:

$$e^x - \epsilon = e^x - \frac{\epsilon}{2} - \frac{\epsilon}{2} \leq s_m(x) - \frac{\epsilon}{2} \leq s'_m(x,n) \leq t_n(x) \leq e^x.$$

□