

## Term Test II

Key.

Suppose  $\nabla f(x, y) = (0, 0)$  throughout  $x^2 + y^2 < r^2$   
and suppose  $x_0^2 + y_0^2 < r^2$  and  $x_1^2 + y_1^2 < r^2$  where  
 $f(x_0, y_0) \neq f(x_1, y_1)$ . Consider  $h(t) = f(t(x_1, y_1) + (1-t)(x_0, y_0))$   
 $= f(x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0))$ , on the line  
segment joining  $(x_0, y_0)$  to  $(x_1, y_1)$ . The entire segment  
lies within the disk and we have  $h(0) = f(x_0, y_0)$

$$\neq f(x_1, y_1) = h(1).$$

By the

Chain rule  $h$  is differentiable on  $(0, 1)$  and continuous  
on  $[0, 1]$  so  $h(1) - h(0) = h'(t) \cdot (1-0)$  for some

$$0 < t < 1. \quad \text{But } h'(t) = D_{\underline{u}} f(t) = \nabla f(x_0 + t(x_1 - x_0),$$

$$y_0 + t(y_1 - y_0)) \cdot \underline{u} = 0 \text{ where } \underline{u} \text{ is the unit vector } \frac{(x_1 - x_0, y_1 - y_0)}{a}$$

$$\text{where } a = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}. \quad \text{This is a contradiction,}$$

$$\text{since } h(1) \neq h(0).$$

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$$2. \quad F(x, y, z, u, v) = xe^y + uz - \cos v - 2$$

$$G(x, y, z, u, v) = u \cos y + x^2 v - yz^2 - 1$$

$$\frac{\partial(F, G)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{bmatrix}$$

$$= \det \begin{bmatrix} z & \sin v \\ \cos y & x^2 \end{bmatrix} = x^2 z - \cos y \sin v$$

$$\text{At } (2, 0, 1, 1, 0) \quad \frac{\partial(F, G)}{\partial(u, v)} = 4 \cdot 1 - \cos 0 \sin 0 = 4 \neq 0$$

so that  $(u, v)$  is a function of  $(x, y, z)$  in a neighborhood

$$\text{Moreover } \left( \frac{\partial u}{\partial z} \right)_{x, y} = \frac{-\det \begin{bmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial v} \end{bmatrix}}{\frac{\partial(F, G)}{\partial(u, v)}} = \frac{-\det \begin{bmatrix} u & \sin v \\ -2yz & x^2 \end{bmatrix}}{x^2 z - \cos y \sin v}$$

$$= - \frac{(ux^2 + 2yz \sin v)}{x^2 z - \cos y \sin v}$$

$$\text{At } (2, 0, 1, 1, 0) \text{ this equals } - \frac{(1 \cdot 4 + 2 \cdot 0 \cdot 1 \cdot \sin 0)}{4 \cdot 1 - \cos 0 \sin 0} = - \frac{4}{4} = -1$$

$$f(x, y, z) = x^2 + y^2 + z^2$$

For

$$G(x, y, z) = x^2 + y^2 - z^2 = 0$$

$$H(x, y, z) = x - 2z - 3 = 0$$

$$\nabla f = (2x, 2y, 2z)$$

$$\nabla G = (2x, 2y, -2z)$$

$$\nabla H = (1, 0, -2)$$

Expressing  $\nabla f = \lambda \nabla G + \mu \nabla H$

$$2x = 2\lambda x + \mu \quad (1)$$

$$2y = 2\lambda y + 0 \quad (2)$$

$$2z = -2\lambda z - 2\mu \quad (3)$$

$$x^2 + y^2 = z^2 \quad (4)$$

and  $x - 2z = 3 \quad (5)$

$$2x(1-\lambda) = \mu \quad (6)$$

$$2y(1-\lambda) = 0 \quad (7)$$

$$-z(1+\lambda) = -\mu \quad (8)$$

$$x^2 + y^2 = z^2 \quad (9)$$

$$x - 2z = 3 \quad (10)$$

From 7  $y=0$  or  $\lambda=1$

Now if  $\lambda=1$ , from 6  $\mu=0$  and from 8  $z=0$

Now (9) implies  $x=y=z=0$ , contradicting  $x-2z=3$ .

So  $\lambda \neq 1$  and  $y=0$ .

-4-

$$2x(1-\lambda) = -(1+\lambda)z \quad \text{from 6 and 8}$$

$$x^2 = z^2 \quad \text{from 9 and } y=0$$

$$\text{So } x = -3, y = 0, z = -3 \quad \text{gives } x - 2x = 3 \quad \lambda = 3, \mu = 12$$

is a critical point and  $f(-3, 0, -3) = 9 + 0 + 9 = 18$

$$\text{For } x = -3 \text{ we get } x + 2x = 3, \quad x = \frac{3}{3} = 1$$

$$y = 0, z = -1, \quad z(1-\lambda) = +(1+\lambda) \quad \mu = \frac{4}{3}$$

$$2 - 2\lambda = +1 + \lambda$$

$$\frac{1}{3} = \lambda$$

$$\text{and } f(1, 0, -1) = 1 + 1 - 2$$

So the max is 18 and the min is 2

4. From the Taylor formula for degree 1 with hierarchy  $k=0$   
 From the Taylor estimated error term

$$f(x+h, t) = f(x, t) + f_1(x, t)h + f_2(x, t) \cdot 0$$

$$+ \frac{1}{2} \left( f_{11}(x+\theta h, t) h^2 \right. \\ \left. + 2f_{12}(x+\theta h, t) h \cdot 0 \right. \\ \left. + f_{22}(x+\theta h, t) \cdot 0^2 \right)$$

$$\text{So } f(x+h, t) - f(x, t) - f_1(x, t)h = \frac{1}{2} f_{11}(x+\theta h, t) h^2$$

$$\text{and } \left| f(x+h, t) - f(x, t) - f_1(x, t)h \right| \leq \left| \frac{1}{2} f_{11}(x+\theta h, t) h^2 \right| \leq \frac{1}{2} g(t) h^2$$

$$\text{Remainder} \left| \frac{F(x+h) - F(x)}{h} - \int_a^b f_1(x, t) dt \right|$$

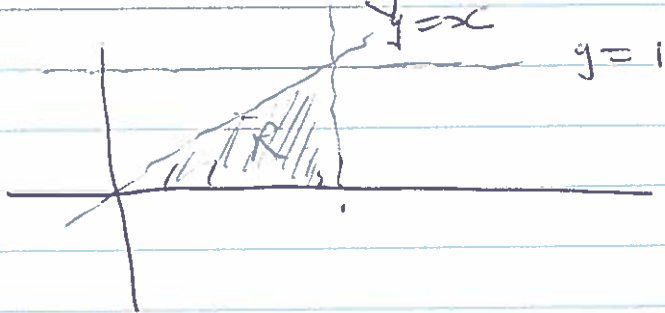
$$\leq \frac{|h|}{2} \int_a^b g(t) dt \leq \frac{|h|}{2} K$$

which tends to 0 as  $h \rightarrow 0$ .

$$\text{So } F'(x) = \int_a^b f_1(x, t) dt \text{ as}$$

desired.

$$\text{Ex. I } \int_0^1 \left( \int_0^1 e^{-x^2} dx \right) dy = \iint_R e^{-x^2} dA$$



$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{-x^2} dy dx$$

$$= \int_{x=0}^{x=1} \left( ye^{-x^2} \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_{x=0}^{x=1} (xe^{-x^2} - 0) dx$$

$$= \left. \frac{-e^{-x^2}}{2} \right|_0^1 = \frac{-e^{-1}}{2} + \frac{e^0}{2}$$

$$= \frac{1}{2} \left( 1 - \frac{1}{e} \right)$$