

Math 381
Assignment 4 - Solutions

13.2 Problem 18

Let $F(x, y, z) = 2x + 3y + 4z$ subject to

$$x \geq 0, y \geq 0, z \geq 0, x + y \geq 2, y + z \geq 2, x + z \geq 2.$$

Since $\nabla F = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ the minimum must

occur at a vertex which is extremal in the

direction $-\nabla F$. So we do not need to be concerned

that the region is unbounded. The region lies in

the first octant, to the right of the ^{vertical} plane

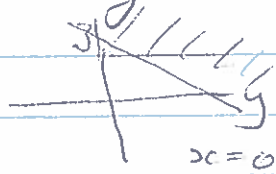
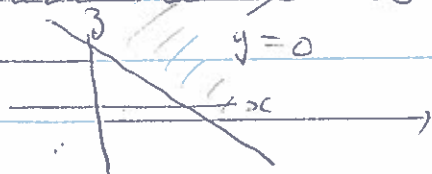
through the line $z=0, x+y=2$ passing through $(2, 0, 0)$

and $(0, 2, 0)$. Note that neither $(2, 0, 0)$ nor $(0, 2, 0)$

are vertices of the polyhedron since they fail to satisfy

$y+z=2$ and $x+z=2$ respectively. The region lies

above the planes $x+z=2$ and $y+z=2$.



There are six boundary planes which give rise to $6C_3 = 20$ candidate vertices, if one wants to do an exhaustive search. Note that setting any one of $x, y, \text{ or } z = 0$ yields the other two taking the value at least 2 (from $x+y \geq 2, y+z \geq 2$ and $x+z \geq 2$) and leads to a contradiction. So $x=0, y=0, z=0$ are not defining planes for a feasible vertex. This leads us to solve $x+y=2, y+z=2, x+z=2$ leading to $x=1, y=1, z=1$ and $F(x, y, z) = 9$.

13.3 Problem 10 Consider $F(x, y, z) = x + 2y - 3z$ over the ellipsoid $\mathcal{E}: x^2 + 4y^2 + 9z^2 \leq 108$.

Because \mathcal{E} is closed and bounded F achieves a minimum and a maximum, but, since $\nabla F = (1, 2, -3) \neq \underline{0}$ this must occur on the boundary $G(x, y, z) = x^2 + 4y^2 + 9z^2 - 108 = 0$.

So we introduce a Lagrange multiplier

λ and set

$$\nabla F = \lambda \nabla G$$

$$(1, 2, -3) = \lambda (2x, 4y, 6z)$$

$$= \mu (x, 2y, 3z) \text{ where}$$

$$\mu = 2\lambda.$$

$$1 = \mu x$$

$$2 = 2\mu y$$

$$-3 = 3\mu z$$

$$\text{and } x^2 + y^2 + z^2 = 108,$$

Thus $\mu, x, y, z \neq 0$ and $x = \frac{1}{\mu}, y = \frac{1}{2\mu}, z = -\frac{1}{3\mu}$

$$\text{so } \frac{1}{\mu^2} + \frac{4}{4\mu^2} + \frac{9}{9\mu^2} = 108 \text{ and } \frac{3}{\mu^2} = 108$$

$$\mu^2 = \frac{1}{36}, \mu = \pm \frac{1}{6}, \text{ This gives}$$

$$(x, y, z) = (6, 12, -18) \text{ and } (x, y, z) = (-6, -12, 18)$$

$$F(6, 12, -18) = 6 + 24 - 54 = -24, \text{ min}$$

$$F(-6, -12, 18) = -6 - 24 + 54 = 24, \text{ max.}$$

B, 3 Problem 12 Let $f(x, y, z) = x^2 + y^2 + z^2$

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subject to $G(x,y,z) = x^2 + y^2 - z^2 = 0$ and

$$H(x,y,z) = x - 2z - 3 = 0.$$

Introducing Lagrange Multiplier λ, μ

and setting $\nabla f = \lambda \nabla G + \mu \nabla H$

we obtain

$$(2x, 2y, 2z) = \lambda(2x, 2y, -2z) + \mu(1, 0, -2)$$

and $x^2 + y^2 - z^2 = 0, \quad x - 2z - 3 = 0$

$$2x = 2\lambda x + \mu$$

$$2y = 2\lambda y$$

$$2z = -2\lambda z - 2\mu$$

$$\begin{aligned} x^2 + y^2 &= z^2 \\ x - 2z &= 3 \end{aligned}$$

Case 1 $y = 0, \quad x = \pm z \quad \text{If } x = z, \quad x = z = -3,$

$$\begin{cases} -6 = -6\lambda + \mu \\ -6 = 6\lambda - 2\mu \end{cases} \quad \mu = 12, \lambda = 3$$

Giving $(x,y,z) = (-3, 0, -3)$

If $x = -z \quad x = 1, z = -1, \mu = \frac{4}{3}, \lambda = \frac{1}{3}$

giving $(x,y,z) = (1, 0, -1)$

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Case 2 $y \neq 0$ Then $\lambda = 1$ $\mu = 0$, $\nu = 0$

$x = 3$, which is impossible for $x^2 + y^2 = z^2$.

Now $f(-3, 0, -3) = 9 + 0 + 9 = 18$ max

$f(1, 0, -1) = 1 + 0 + 1 = 2$ min.

13.3 Problem 22 Let $f(x, y, z) = xy + z^2$

on the ball $x^2 + y^2 + z^2 \leq 1$.

For interior points set $\nabla f = (y, x, 2z) = \underline{0}$

giving the origin $(0, 0, 0)$.

For the boundary $G(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

and we set $\nabla f = \lambda \nabla G$, $G = 0$

giving

$$y = 2\lambda x$$

$$x = 2\lambda y$$

$$2z = 2\lambda z$$

$$x^2 + y^2 + z^2 = 1$$

If $z \neq 0$, $\lambda = 1$, $x = 2y$, $2x = y \Rightarrow x = y = 0$,

$z = \pm 1$. This gives $(0, 0, 1)$ and $(0, 0, -1)$.

If $z=0$ we cannot have $x=0$ or $y=0$

(or both are given $x^2+y^2+z^2=0$) $y = 2\lambda x = 2\lambda z, \lambda y$

gives $y = 4\lambda^2 y$ so $\lambda = \pm \frac{1}{2}$. This gives $x=y$

or $x=-y$. From $x^2+y^2+0^2=1$ we get

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$\text{and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

Calculating $f(x,y,z) = xy + z^2$

$$f(0,0,0) = 0$$

$$f(0,0,1) = 1 = f(0,0,-1)$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{2} = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

So the minimum value is $-\frac{1}{2}$ and the maximum is 1.

15.4 Problem 12

$$\text{let } y = p + qx$$

Find the

selection of p & q that give the best least squares fit

to $(1, 0.11), (2, 1.62), (3, 4.07), (4, 7.55), (6, 17.63)$

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and (7, 24, 20)

$$\text{minimize } E = \sum (y_i - px_i - q)^2$$

$$\frac{\partial E}{\partial p} = -2 \sum (y_i - px_i - q)x_i$$

$$\frac{\partial E}{\partial q} = -2 \sum (y_i - px_i - q) \cdot 1$$

Setting $\frac{\partial E}{\partial p} = 0 = \frac{\partial E}{\partial q}$ we get

$$p \sum x_i^2 + q \sum x_i = \sum y_i x_i$$

$$p \sum x_i + q \sum 1 = \sum y_i$$

$$\text{Now } \sum 1 = 6, \quad \sum x_i = 1+2+3+4+6+7 = 23$$

$$\sum x_i y_i = 1 \cdot 0.11 + 2 \cdot 1.62 + 3 \cdot 4.07 + 4 \cdot 7.55 +$$

$$6 \cdot 17.63 + 7 \cdot 24.20 = 320.94$$

$$\sum y_i = 0.11 + 1.62 + 4.07 + 7.55 + 17.63 + 24.20$$

$$= 55.18$$

$$\sum x_i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 7^2 = 115$$

So we have

$$115p + 23q = 320.94$$

$$23p + 6q = \dots - 24.20$$

$$p = \frac{1369.04}{161} = 8.503354 \text{ and } q = \frac{-2759.72}{161} = -28.562856$$

161 The predicted value for $x=5$ is $5 \cdot \frac{1369.04}{161} - \frac{2759.72}{161}$

13.4 Problem 18. Consider $F(a, b, c) = \int_0^1 (x^3 - ax^2 - bx - c)^2 dx$

$$\frac{\partial F}{\partial a} = \int_0^1 2(x^3 - ax^2 - bx - c)(-x^2) dx$$

$$\frac{\partial F}{\partial b} = \int_0^1 2(x^3 - ax^2 - bx - c)(-x) dx$$

$$\frac{\partial F}{\partial c} = \int_0^1 2(x^3 - ax^2 - bx - c)(-1) dx$$

Setting all three to 0 gives

$$a \int_0^1 x^4 dx + b \int_0^1 x^3 dx + c \int_0^1 x^2 dx = \int_0^1 x^5 dx$$

$$a \int_0^1 x^3 dx + b \int_0^1 x^2 dx + c \int_0^1 x dx = \int_0^1 x^4 dx$$

$$a \int_0^1 x^2 dx + b \int_0^1 x dx + c \int_0^1 1 dx = \int_0^1 x^3 dx$$

So

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{5} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{5} \\ \frac{1}{4} \end{bmatrix}$$

13.5 Problem 6 Let $F(x) = \int_0^{\infty} e^{-xt} \frac{\sin t}{t} dt$

Note that $\frac{\partial}{\partial x} \left(e^{-xt} \frac{\sin t}{t} \right) = -e^{-xt} \sin t$

So $\int_0^{\infty} e^{-xt} \sin t dt = \frac{d}{dx} \int_0^{\infty} -e^{-xt} \frac{\sin t}{t} dt$

By Problem 15 $\int_0^{\infty} e^{-xt} \sin t dt = \frac{1}{1+x^2}$ for $x > 0$

and so $\frac{d}{dx} \int_0^{\infty} e^{-xt} \frac{\sin t}{t} dt = -\frac{1}{1+x^2}$

and $F(x) = \int_0^{\infty} e^{-xt} \frac{\sin t}{t} dt = \int -\frac{1}{1+x^2} dx = -\text{Arctan } x + C$

Now $\lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0$ so $\left| e^{-xt} \frac{\sin t}{t} \right| \leq K e^{-xt}$

And $\int_0^N e^{-xt} dt = \frac{-e^{-xt}}{x} \Big|_0^N = \frac{-e^{-xN}}{x} + \frac{e^0}{x} \rightarrow \frac{1}{x}$

as $N \rightarrow \infty$. So $\lim_{x \rightarrow \infty} F(x) = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

$$\text{Now } \lim_{x \rightarrow \infty} \text{Arctan } x = \frac{\pi}{2}, \quad \text{So } 0 = -\frac{\pi}{2} + C$$

$$\text{and } C = \frac{\pi}{2}$$

$$\text{So } F(x) = \frac{\pi - \text{Arctan } x}{2}$$

$$\text{Now } \int_0^{\infty} \frac{\sin t}{t} dt = \lim_{x \rightarrow 0} \int_0^{\infty} e^{-xt} \frac{\sin t}{t} dt = \lim_{x \rightarrow 0} \frac{\pi - \text{Arctan } x}{2} = \frac{\pi}{2}$$

13.5 Problem 20 Consider the family of curves

parameterized by c , for fixed k given by

$$x^2 + (y-c)^2 = kc^2$$

$$\text{let } f(x, y, c) = x^2 + (y-c)^2 - kc^2$$

$$\frac{\partial f}{\partial c} = -2(y-c) - 2kc = 2(1-k)c - 2y$$

$$\text{Setting } f=0, \quad \frac{\partial f}{\partial c} = 0$$

$$\frac{\partial f}{\partial c} = 0 \text{ gives } c = \frac{y}{1-k}, \quad k \neq 1$$

$$x^2 + ((1-k)c)^2 = kc^2$$

$$x^2 + k^2 c^2 = kc^2$$

$k \neq 1$

$$x^2 = c^2 = \frac{y^2}{(1-k)^2} \quad y = \pm \frac{1+k}{k} x$$

$$\pm ky = (1-k)x$$

So for $k \neq 0, 1$ the envelope

is a pair of lines.

14.1 Problem 8 Let D be the disk $x^2 + y^2 \leq 25$

and P the partition of $-5 \leq x \leq 5, -5 \leq y \leq 5$ into ~~100~~

1x1 unit subboxes. Let $f(x, y) = \begin{cases} 1 & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$

Let (x_{ij}^*, y_{ij}^*) be the corner of the i -th j -th

square furthest from the origin. We have $3^2 + 4^2 = 5^2$

so there are 5 squares in the bottom left corner first

column, 2 in the second column and one each in the

third fourth and fifth where the furthest corner

lies outside the circle ($5^2 + 1^2 > 5^2$), for a

total of 9 in each quadrant, and 36 in total,

where $f(x_{ij}^*, y_{ij}^*) = 0$. So the Riemann

sum is $100 - 36 = 64$.

14.2 Problem 2

$$\iint_R (xy + y^2) dA = \int_0^1 \int_0^y (xy + y^2) dx dy = I$$

where R is the triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$

$$I = \int_0^1 \left. \frac{x^2 y}{2} + xy^2 \right|_{x=0}^{x=y} dy = \int_0^1 \left(\frac{y^3}{2} + y^3 \right) dy$$

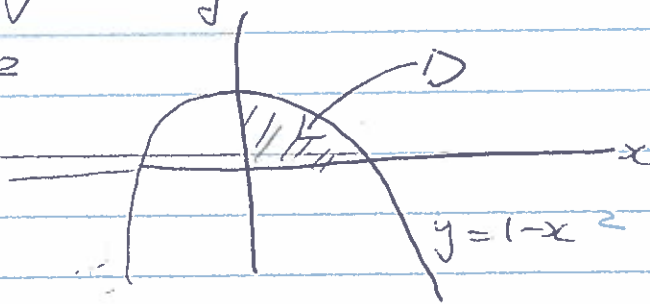
$$= \int_0^1 \frac{3}{2} y^3 dy = \frac{3}{8} y^4 \Big|_0^1 = \frac{3}{8} - 0 = \frac{3}{8}$$

14.2 Problem 10

let D be the finite region

in the first quadrant bounded by the coordinate

axes and the curve $y = 1 - x^2$



$$\iint_D x \cos y dA$$

$$= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x^2} x \cos y dy \right) dx = \int_{x=0}^{x=1} x \sin y \Big|_{y=0}^{y=1-x^2} dx$$

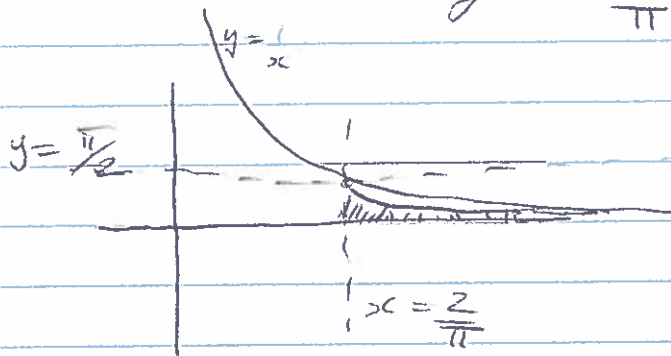
$$= \int_{x=0}^{x=1} (x \sin(1-x^2) - 0) dx = \frac{1}{2} \cos(1-x^2) \Big|_{x=0}^{x=1}$$

$$= \frac{1}{2} \cos(1-1^2) - \frac{1}{2} \cos(1-0^2)$$

$$= \frac{1}{2} (\cos 0 - \cos 1) = \frac{1}{2} (1 - \cos 1).$$

14.3 Problem 12 Consider $\iint_R \frac{1}{x} \sin \frac{1}{x} dA$

where R is the region $\frac{2}{\pi} \leq x < \infty$, $0 \leq y \leq \frac{1}{x}$



Note $0 < \sin \frac{1}{x} \leq 1$ on $[\frac{2}{\pi}, \infty)$.

We look at $\lim_{N \rightarrow \infty} \int_{x=\frac{2}{\pi}}^N \left(\int_{y=0}^{\frac{1}{x}} \frac{1}{x} \sin \frac{1}{x} dy \right) dx$

$$= \lim_{N \rightarrow \infty} \int_{x=\frac{2}{\pi}}^N \left. \frac{y}{x} \sin \frac{1}{x} \right|_{y=0}^{y=\frac{1}{x}} dx$$

$$= \lim_{N \rightarrow \infty} \int_{x=\frac{2}{\pi}}^N \frac{1}{x^2} \sin \frac{1}{x} dx \leq \lim_{N \rightarrow \infty} \int_{x=\frac{2}{\pi}}^N \frac{1}{x^2} dx$$

$$= \lim_{N \rightarrow \infty} \left. -\frac{1}{x} \right|_{x=\frac{2}{\pi}}^N = -\frac{1}{N} + \frac{1}{\frac{2}{\pi}} = \frac{2}{\pi} - \frac{1}{N} \rightarrow \frac{2}{\pi}$$

So $\iint_R \frac{1}{x} \sin \frac{1}{x} dA$ converges.

Also

$$\lim_{N \rightarrow \infty} \int_{\frac{\pi}{2}}^{x=N} \frac{1}{x^2} \sin \frac{1}{x} dx = \lim_{N \rightarrow \infty} \cos \frac{1}{x} \Big|_{x=\frac{\pi}{2}}^{x=N}$$

$$= \lim_{N \rightarrow \infty} \left(\cos \frac{1}{N} - \cos \frac{\pi}{2} \right)$$

$$= \cos 0 = 1$$

So $\iint_R \frac{1}{x} \sin \frac{1}{x} dA = 1$.

14.4 Problem 2 Let D be the disk

$$x^2 + y^2 \leq a^2 \text{ with } a > 0$$

$$\iint_D \sqrt{x^2 + y^2} dA = \int_{\theta=0}^{2\pi} \int_{r=0}^a r \cdot r dr d\theta = \left(\int_0^{2\pi} 1 d\theta \right) \left(\int_0^a r^2 dr \right)$$

$$= 2\pi \cdot \frac{a^3}{3} = \frac{2\pi a^3}{3}$$

14.4 Problem 8 Let Q be the quarter disk given by

$$x \geq 0, y \geq 0, x^2 + y^2 \leq a^2, \text{ with } a > 0.$$

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$$\begin{aligned}
 \iint_Q (x+y) dA &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=a} (r\cos\theta + r\sin\theta) r dr d\theta \\
 &= \left(\int_{\theta=0}^{\theta=\pi/2} (\cos\theta + \sin\theta) d\theta \right) \left(\int_{r=0}^{r=a} r^2 dr \right) \\
 &= (\sin\theta - \cos\theta) \Big|_{\theta=0}^{\theta=\pi/2} \cdot \frac{a^3}{3} \\
 &= (1 - (-1)) \frac{a^3}{3} = \frac{2a^3}{3}
 \end{aligned}$$