Math 381 KEY

Assignment II

12.1 Problem 24 $f(x,y) = \dfrac{y}{x^2+y^2}$ $(x,y) \neq (0,0)$

level curve $f(x,y)=0 \Rightarrow y=0$, Punctured

x-axis 

level curve $f(x,y)=k \neq 0$ Set $\dfrac{1}{k} = l$

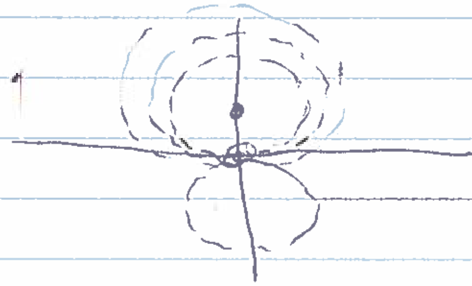
Then $\dfrac{y}{x^2+y^2} = \dfrac{1}{l}$ and $ly = x^2+y^2$

whence $x^2+y^2-ly=0$ so $x^2+\left(y-\frac{l}{2}\right)^2 = \dfrac{l^2}{4}$

which is a circle of radius $\dfrac{|l|}{2} = \dfrac{1}{|2k|}$ and centre

at $\left(0, \frac{1}{2k}\right)$.

1 STYLE



Problem 36 The level surfaces of $f(x,y,z) = C$ is

a plane having intercepts $C^3, 2C^3, 3C^3$ on the

x-axis, y-axis and z-axis respectively. An equation

of the plane is $6x+3y+2z = 6C^3$, so $f(x,y,z) = \left(\dfrac{6x+3y+2z}{6}\right)^{\frac{1}{3}}$

§12.2 Problem 12 Consider $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$

Along the coordinate axes

$y=0$ $\lim_{x\to 0} \frac{x^2\cdot 0}{2x^4+0} = 0$

and $x=0$ $\lim_{y\to 0} \frac{0\cdot y^2}{2\cdot 0+y^4} = 0$

Along the line $y=kx$, $k\neq 0$ $x\neq 0$

$$\lim_{x\to 0} \frac{x^2(kx)^2}{2x^4+(kx)^4} = \lim_{x\to 0} \frac{k^2}{2+k^4} \neq 0,$$

So $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$ does not exist.

Problem 14 Consider $f(x,y) = \frac{x^3-y^3}{x-y}$ $x\neq y$.

Now $x^3-y^3 = (x-y)(x^2+xy+y^2)$

so for $x\neq y$ $f(x,y) = x^2+xy+y^2$

If we define $f(x,y) = x^2+xy+y^2$ when $x=y$,

then f is continuous on R^2.

§12.3 Problem 12 Consider $f(x,y) = \begin{cases} \dfrac{x^2-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

For $D_1 f(0,0)$ consider $\lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{h^2-2\cdot 0^2}{h-0} - 0}{h} = 1$$

For $D_2 f(0,0)$ consider $\lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{\frac{0^2-2k^2}{0-k} - 0}{k} = -1$$

Problem 16. Let $f(x,y) = e^{xy}$

Then $f_1(x,y) = ye^{xy}$ and $f_2(x,y) = xe^{xy}$

At $(2,0)$

$f(2,0) = e^0 = 1$
$f_1(2,0) = 0e^0 = 0$
$f_2(2,0) = 2e^0 = 2$

So a normal to the tangent plane is $(0,2,-1)$

An equation of the normal line is $(x,y,z) = (2,0,1) + t(0,2,-1)$

and an equation of the tangent plane is $z = 1 + 0(x-2) + 2(y-0)$
or $z = 1 + 2y$

Problem 24. Consider the surface with equation

$$z = xy\,e^{-(x^2+y^2)/2}$$

Then $\dfrac{\partial z}{\partial x} = y\,e^{-(x^2+y^2)/2} - \cancel{2}x^2y\,e^{-(x^2+y^2)/2}$

and $\dfrac{\partial z}{\partial y} = x\,e^{-(x^2+y^2)/2} - \cancel{2}xy^2\,e^{-(x^2+y^2)/2}$

A normal direction to the tangent plane at (x,y,z) is $(f_1, f_2, -1)$ or $\left(y e^{-(x^2+y^2)/2}(1-2x^2),\ x e^{-(x^2+y^2)/2}(1-2y^2),\ -1\right)$

For the tangent plane to be horizontal the normal must be parallel to $(0, 0, 1)$ so we get

$$y\,e^{-(x^2+y^2)/2}(1-2x^2) = 0$$

$$x\,e^{-(x^2+y^2)/2}(1-2y^2) = 0$$

Since $e^{-(x^2+y^2)/2} \neq 0$ we see that

$$y(1-2x^2) = 0$$

and

$$x(1-2y^2) = 0$$

This gives $(x,y) = (0,0)$, $(x,y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

Problem 36, Let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Along $x=0$ $\lim\limits_{y\to 0} f(x,y) = \lim\limits_{y\to 0} \dfrac{2\cdot 0\cdot y}{0^2+y^2} = 0$

$= \lim\limits_{x\to 0} f(x,y) = \lim\limits_{x\to 0} \dfrac{2\cdot x\cdot 0}{x^2+0^2} = 0$

But along $y=kx$ $k\neq 0$

$\lim\limits_{x\to 0} f(x,kx) = \lim\limits_{x\to 0} \dfrac{2\cdot x\cdot kx}{x^2+k^2x^2}$

$= \lim\limits_{x\to 0} \dfrac{2k}{1+k^2} \neq 0,$

So f is not continuous at $(0,0)$.

To calculate $f_1(0,0)$ consider

$\lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{2\cdot h\cdot 0}{h^2+0^2}}{h} = 0$

and $\lim\limits_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{\frac{2\cdot 0\cdot k}{0^2+k^2}}{k} = 0$

Shows $f_1(0,0) = 0 = f_2(0,0)$.

Problem 4

§ 12.4 Let $f(x,y) = z = \sqrt{3x^2+y^2}$

Then $\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{3x^2+y^2}} \cdot 6x = \dfrac{3x}{\sqrt{3x^2+y^2}}$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{3x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{3x^2+y^2}}$$

So $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{3}{\sqrt{3x^2+y^2}} + 3x\left(\dfrac{-1}{2}\right)\dfrac{6x}{(3x^2+y^2)^{3/2}}$

$$= \frac{3(3x^2+y^2) - 9x^2}{(3x^2+y^2)^{3/2}} = \frac{3y^2}{(3x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-1}{2} \cdot \frac{3x}{(3x^2+y^2)^{3/2}} \cdot 2y = \frac{-3xy}{(3x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{\sqrt{3x^2+y^2}} - \frac{1}{2}y\frac{2y}{(3x^2+y^2)^{3/2}}$$

$$= \frac{3x^2+y^2-y^2}{(3x^2+y^2)^{3/2}} = \frac{3x^2}{(3x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{2}\frac{y}{(3x^2+y^2)^{3/2}} \cdot 6x = \frac{-3xy}{(3x^2+y^2)^{3/2}}$$

so $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

Problem 10 Let $z = f(x,y) = \dfrac{x}{x^2+y^2}$, $(x,y) \neq (0,0)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} - \frac{x}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2x}{(x^2+y^2)^2} + (y^2-x^2)(-2)\frac{2x}{(x^2+y^2)^3}$$

$$= \frac{-2x(x^2+y^2)-4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2x}{(x^2+y^2)^2} - 2xy\,(-2)\frac{2y}{(x^2+y^2)^3}$$

$$= \frac{-2x(x^2+y^2)+8xy^2}{(x^2+y^2)^3} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2x^3-6xy^2}{(x^2+y^2)^3} + \frac{-2x^3+6xy^2}{(x^2+y^2)^3} = 0$$

for $(x,y) \neq (0,0)$, as required.

Problem 16

Let
$$F(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$

$$F_1(x,y) = \frac{2y(x^2-y^2)+4x^2y}{(x^2+y^2)} - \frac{2xy(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= \frac{(6x^2y-2y^3)(x^2+y^2) - 4x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{6x^4y+6x^2y^3-2x^2y^3-2y^5-4x^4y+4x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{2x^4y+8x^2y^3-2y^5}{(x^2+y^2)^2}$$

$$F_2(x,y) = \frac{2x(x^2-y^2)-4xy^2}{x^2+y^2} - \frac{2xy(x^2-y^2)2y}{(x^2+y^2)^2}$$

$$= \frac{(2x^3-6xy^2)(x^2+y^2) - 4xy^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{2x^5+2x^3y^2-6x^3y^2-6xy^4-4x^3y^2+4xy^4}{(x^2+y^2)^2}$$

$$= \frac{2x^5 - 8x^3y^2 - 2xy^4}{(x^2+y^2)^2}$$

$$F_{11}(x,y) = \frac{8x^3y + 16xy^3}{(x^2+y^2)^2} - \frac{2(2x^4y + 8x^2y^3 - 2y^5)2x}{(x^2+y^2)^3}$$

$$= \frac{(8x^3y + 16xy^3)(x^2+y^2) - 8x^5y - 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{8x^5y + 8x^3y^3 + 16x^3y^3 + 16xy^5 - 8x^5y - 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{-8x^3y^3 + 24xy^5}{(x^2+y^2)^3}$$

$$F_{12}(x,y) = \frac{2x^4 + 24x^2y^2 - 10y^4}{(x^2+y^2)^2} - \frac{2(2x^4y + 8x^2y^3 - 2y^5)2y}{(x^2+y^2)^3}$$

$$= \frac{(2x^4 + 24x^2y^2 - 10y^4)(x^2+y^2) - 8x^4y^2 - 32x^2y^4 + 8y^6}{(x^2+y^2)^3}$$

$$= \frac{2x^6 + 24x^4y^2 - 10x^2y^4 + 2x^4y^2 + 24x^2y^4 - 10y^6 - 8x^4y^2 - 32x^2y^4 + 8y^6}{(x^2+y^2)^3}$$

$$= \frac{2x^6 + 18x^4y^2 - 18x^2y^4 - 2y^6}{(x^2+y^2)^3}$$

$$F_{22}(x,y) = \frac{-16x^3y - 8xy^3}{(x^2+y^2)^2} - \frac{2(2x^5 - 8x^3y^2 - 2xy^4)(2y)}{(x^2+y^2)^3}$$

$$= \frac{(-16x^3y - 8xy^3)(x^2+y^2) - 8x^5y + 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{-16x^5y - 16x^3y^3 - 8x^3y^3 - 8xy^5 - 8x^5y + 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{-24x^5y + 8x^3y^3}{(x^2+y^2)^3}$$

$$F_{21}(x,y) = \frac{10x^4 - 24x^2y^2 - 2y^4}{(x^2+y^2)^2} - \frac{2(2x^5 - 8x^3y^2 - 2xy^4)\,2x}{(x^2+y^2)^3}$$

$$= \frac{(10x^4 - 24x^2y^2 - 2y^4)(x^2+y^2) - 8x^6 + 32x^4y^2 + 8x^2y^4}{(x^2+y^2)^3}$$

$$= \frac{10x^6 + 10x^4y^2 - 24x^4y^2 - 24x^2y^4 - 2x^2y^4 - 2y^6 - 8x^6 + 32x^4y^2 + 8x^2y^4}{(x^2+y^2)^3}$$

$$= \frac{2x^6 + 18x^4y^2 - 18x^2y^4 - 2y^6}{(x^2+y^2)^3}$$

At $(0,0)$

$$F_1(0,0) = \lim_{h \to 0} \frac{F(0+h,\,0) - F(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

and
$$F_2(0,0) = \lim_{k \to 0} \frac{F(0,\,0+k) - F(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$F_{12}(0,0) = \lim_{k \to 0} \frac{F_1(0,\,0+k) - F_1(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\dfrac{-2k^5}{(0+k^2)^2} - 0}{k} = -2$$

$$= \lim_{k \to 0} \frac{-2k^5}{k^5} = -2$$

$$F_{21}(0,0) = \lim_{h \to 0} \frac{F_2(0+h,\,0) - F_2(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{2h^5}{(h^2+0)^2} - 0}{h} = \lim_{h \to 0} \frac{2h^5}{h^5} = 2$$

The conditions to check are that $F_1(x,y)$ $F_2(x,y)$ are continuous in a neighborhood of $(0,0)$ and that F_{12} and F_{21} are continuous at $(0,0)$.

F_1 and F_2 as well as F_{12}, F_{21} are continuous when $(x,y) \neq (0,0)$ so the problem is to analyze their behavior at $(0,0)$.

Consider $F_1(x,y) = y \dfrac{2x^4 + 8x^2y^2 - 2y^4}{(x^2+y^2)^2}$

Now $\left| \dfrac{2x^4 + 8x^2y^2 - 2y^4}{(x^2+y^2)^2} \right| \leq 4$

So $\lim\limits_{(x,y)\to(0,0)} F_1(x,y) = 0 = F_1(0,0)$

Also $F_2(x,y) = x \dfrac{2x^4 - 8x^2y^2 - 2y^4}{(x^2+y^2)^2}$

and similarly $\lim\limits_{(x,y)\to(0,0)} F_2(x,y) = 0$.

However $\lim\limits_{k\to 0} F_{12}(0,k) = \lim\limits_{k\to 0} \dfrac{-2k^6}{(0^2+k^2)^3} = -2$

while $\lim\limits_{h\to 0} F_{12}(h,0) = \lim\limits_{h\to 0} \dfrac{2h^6}{(h^2+0^2)^3} = 2$ so F_{12} is not continuous at $(0,0)$, so there is no contradiction,

§ 12.5 Problems 16

Suppose that f is a harmonic function of two variables, that is $f_{11} + f_{22} = 0$. Let $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

Note that u is harmonic (except at $0,0$)

Since $\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$, $\frac{\partial^2 u}{\partial x^2} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial y^2} = \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}.$$

Similarly v is harmonic, that is $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Now let $w = f(u, v)$

Then $\frac{\partial w}{\partial x} = f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x}$

$$\text{So } \frac{\partial^2 w}{\partial x^2} = f_{11}(u,v)\left(\frac{\partial u}{\partial x}\right)^2 + f_{21}(u,v)\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + f_1(u,v)\frac{\partial^2 u}{\partial x^2}$$

$$+ f_{12}(u,v)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_{22}(u,v)\left(\frac{\partial v}{\partial x}\right)^2 + f_2(u,v)\frac{\partial^2 v}{\partial x^2}$$

Now $\frac{\partial w}{\partial y} = f_1(u,v)\frac{\partial u}{\partial y} + f_2(u,v)\frac{\partial v}{\partial y}$

So $$\frac{\partial^2 w}{\partial y^2} = f_{11}(u,v)\left(\frac{\partial u}{\partial y}\right)^2 + f_{21}(u,v)\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + f_1(u,v)\frac{\partial^2 u}{\partial y^2}$$

$$+ f_{12}(u,v)\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_{22}(u,v)\left(\frac{\partial v}{\partial y}\right)^2 + f_2(u,v)\frac{\partial^2 v}{\partial y^2}$$

Now $f_{12}(u,v) = f_{21}(u,v)$

So $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f_{11}(u,v)\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$+ f_{22}(u,v)\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)$$

$$+ 2 f_{12}(u,v)\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$$

$$+ f_1(u,v)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2(u,v)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

Note that $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{y^2 - x^2}{(x^2+y^2)^2}\right)^2 + \left(\frac{-2xy}{(x^2+y^2)^2}\right)^2$$

and $$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{2xy}{(x^2+y^2)^2}\right)^2 + \left(\frac{x^2 - y^2}{(x^2+y^2)^2}\right)^2$$

So $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \left(f_{11}(u,v) + f_{22}(u,v)\right)\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$+ 2 f_{12}(u,v)\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$$

$$+ f_1(u,v)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2(u,v)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

which, using that f, u, v are harmonic reduces

to $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2 f_{12}(u,v)\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$$

But $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}\,\frac{2xy}{(x^2+y^2)^2} + \left(\frac{-2xy}{(x^2+y^2)^2}\right)\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right),$$

$$= 0$$

So $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$ and w is harmonic.

Problem 30 From 29b $F_1(x,y) = -F_2(y,x)$

and $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$

where $F(x,y)$ is the problem of §12.4 Problem 16, discussed above.

Since $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$

we get $F_{12}(x,y) = -F_{12}(y,x)$. Setting $x = y \neq 0$

we read $F_{12}(x,x) = -F_{12}(x,x)$, so $2F_{12}(x,x) = 0$

and $F_{12}(x,x) = 0$ for $x \neq 0$.

Problem 18

$$f(\rho, \phi, \theta) = (x, y, z) \text{ where}$$

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\text{Jacobian} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & +\rho\sin\phi\cos\theta & 0 \end{bmatrix}$$

Det(Jacobian)

$$= \sin\phi\cos\theta\left(\rho^2\sin^2\phi\cos\theta\right) - \sin\phi\sin\theta\left(-\rho^2\sin^2\phi\sin\theta\right)$$

$$+ \cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right)$$

$$= \rho^2\sin\phi\left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\right) + \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$= \rho^2\sin\phi\left(\sin^2\phi + \cos^2\phi\right) = \rho^2\sin\phi$$

From the earlier exercise we calculated that

$F_{12}(0,0) = -2 \neq \lim_{x\to 0} F_{12}(x,x)$, so F_{12} is

not continuous at $(0,0)$.

§ 12.6 Problem 12 (Problem 8 in 6th Edition)

The radius and height of a right circular conical tank

are measured to be 25 ft and 21 ft respectively, each

accurate to $\frac{1}{24}$ ft.

$$V = \frac{1}{3}\pi r^2 h$$

So $$dV = \frac{\partial V}{\partial r}\cdot dr + \frac{\partial V}{\partial h}\cdot dh$$

$$= \frac{2}{3}\pi r h\,dr + \frac{1}{3}\pi r^2\,dh$$

$$dV \approx \frac{2}{3}\pi(25)(21)\left(\frac{1}{24}\right) + \frac{1}{3}\pi(25)^2\left(\frac{1}{24}\right)$$

$$= \frac{25\pi}{3}\left(\frac{42+25}{24}\right) = 25\pi\,\frac{23}{24}$$

$$\sim \underline{75.3 \text{ ft}^3}$$

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