MATHEMATICS 271 MIDTERM November 9, 2009

[6] 1. **Use the Euclidean algorithm** to find $\gcd(81, 25)$. Then use your work to write $\gcd(81, 25)$ in the form $81a + 25b$ where a and b are integers.

Solution. We get

$$81 = 3 \cdot 25 + 6, \qquad (\text{so } 6 = 81 - 3 \cdot 25)$$
$$25 = 4 \cdot 6 + 1, \qquad (\text{so } 1 = 25 - 4 \cdot 6)$$
$$6 = 6 \cdot 1 + 0 \qquad (\text{optional}),$$

so $\gcd(81, 25) = 1$ (the last nonzero remainder). Now reversing these steps gives us

$$\begin{aligned} \gcd(81, 25) = 1 &= 25 - 4 \cdot 6 \\ &= 25 - 4(81 - 3 \cdot 25) = 25 - 4 \cdot 81 + 12 \cdot 25 = 13 \cdot 25 - 4 \cdot 81, \end{aligned}$$

so $a = -4$ and $b = 13$.

Alternately, we could use the table method taught in class, to get:

what the Q-R Thm says	row operation		81	25	what this row says
		81	1	0	$81 = 1 \cdot 81 + 0 \cdot 25$
		25	0	1	$25 = 0 \cdot 81 + 1 \cdot 25$
$81 = 3 \cdot 25 + 6$	$R1 - 3R2$	6	1	-3	$6 = 1 \cdot 81 + (-3) \cdot 25$
$25 = 4 \cdot 6 + 1$	$R2 - 4R3$	1	-4	13	$1 = (-4) \cdot 81 + 13 \cdot 25$

Thus from the last row we see that $\gcd(81, 25) = 1 = (-4) \cdot 81 + 13 \cdot 25$.

[6] 2. Find the number of ways to arrange 5 women and 5 men in a row so that all the men are to the left of all the women. Explain (but do not simplify) your answer.

Solution. There are 5! ways to arrange the five men in a row. For each such arrangement, there are 5! ways to arrange the five women to the right of the five men. Thus by the multiplication rule, there are $5! \cdot 5!$ ways to arrange all ten people.

Aside: Note that $5! + 5!$ is the solution to a different problem, namely: Find the number of ways to choose *either* the five men *or* the five women and arrange them in a row.

3. Let $\mathcal{S}$ be the statement:

for all sets A and B, if $2 \in A$ and $A \cap B = \emptyset$ then $2 \notin B$.

[4] (a) Prove $\mathcal{S}$, using contradiction.

Solution. Let A and B be sets and assume that $2 \in A$ and $A \cap B = \emptyset$. Now to get a contradiction, assume that $2 \in B$. Then since $2 \in A$ and $2 \in B$, we get that $2 \in A \cap B$, which is a contradiction to $A \cap B = \emptyset$. Therefore $2 \in B$ is impossible, so $2 \notin B$, and thus statement $\mathcal{S}$ is true.

Alternatively, we could write out the *negation* of $\mathcal{S}$ and prove in a very similar way that it is false. The negation is:

there exist sets A and B such that $2 \in A$ and $A \cap B = \emptyset$ and also $2 \in B$.

We prove that the negation is false by contradiction. Suppose that it is true, and let A and B be sets so that $2 \in A$, $A \cap B = \emptyset$ and $2 \in B$. Then since $2 \in A$ and $2 \in B$, we get that $2 \in A \cap B$, which is a contradiction to $A \cap B = \emptyset$. Thus the negation must be false, so the original statement $\mathcal{S}$ must be true.

[1 + 4] (b) Write out (as simply as possible) the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

Solution. The converse is

for all sets A and B, if $2 \notin B$ then $2 \in A$ and $A \cap B = \emptyset$.

The converse is **false**. The simplest counterexample is $A = \emptyset$, $B = \emptyset$. Then $2 \notin B$, so the hypothesis of the converse is true, but it is not true that $2 \in A$, so the conclusion "$2 \in A$ and $A \cap B = \emptyset$" fails. There are lots of other counterexamples.

[2 + 1] (c) Write out (as simply as possible) the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.

Solution. The contrapositive is

for all sets A and B, if $2 \in B$ then $2 \notin A$ **or** $A \cap B \neq \emptyset$.

The contrapositive is **true**, because it is equivalent to the original statement $\mathcal{S}$ which is true.

[6] 4. Prove **using mathematical induction** (or well ordering) that $5n - 1 \leq 4^n$ for all integers $n \geq 1$.

Solution. Basis step: When $n = 1$ the statement is $5 \cdot 1 - 1 \leq 4^1$ which is true since both sides are equal to 4.

Inductive step: Assume that $5k - 1 \leq 4^k$ for some integer $k \geq 1$. We want to prove that $5(k+1) - 1 \leq 4^{k+1}$, which can be rewritten as $5k + 4 \leq 4^{k+1}$. By multiplying both sides of the assumption by 4, we get

$$4^{k+1} = 4 \cdot 4^k \geq 4(5k - 1) = 20k - 4.$$

Now **if we could prove that**

$$20k - 4 \geq 5k + 4 \tag{1}$$

for any integer $k \geq 1$, it would follow that

$$4^{k+1} \geq 20k - 4 \geq 5k + 4,$$

and the inductive step would be proven. So we want to prove (1). But (1) is equivalent to $15k \geq 8$ or $k \geq 8/15$, which is true since $k \geq 1$.

[*Note*: Here is a one-line way to prove that $5(k+1)-1 \le 4^{k+1}$ using the inductive hypothesis $5k-1 \le 4^k$ and the fact that $2 \le 4^k$ for all $k \ge 1$:

$$5(k+1)-1 = (5k-1)+2+2+1 \le 4^k + 4^k + 4^k + 4^k = 4 \cdot 4^k = 4^{k+1}.$$

But this method may be harder to find?]

So this proves the inductive step, so $5n-1 \le 4^n$ is true for all integers $n \ge 1$ by induction.

Note that the statement $5n-1 \le 4^n$ is actually true for all integers $n \ge 0$ (since $5 \cdot 0 - 1 \le 4^0$ says $-1 \le 1$ which is true), but to prove this by induction you would need to include **both** $n=0$ and $n=1$ into the basis step, since the inductive step needs $k \ge 1$.