MATH 271 ASSIGNMENT 4

**Due 4:00 PM Friday, November 27, 2009.** Hand your assignment to me in class or in the lab, or in my office (MS566, under the door if I'm not there). Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. Let $n$ be a positive integer.

   (a) Prove that $\sum_{i=1}^{n+1} i\binom{n+1}{i} = \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=2}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}$. [*Hint*: Pascal's Formula (page 360).]

   (b) Use part (a) and induction on $n$ to prove the identity $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$ for all integers $n \geq 1$. [*Hint*: Replace $i-1$ by $j$ in the last two sums in the formula in part (a). You may also use Example 6.7.2 on page 368.]

   (c) Give a *combinatorial* proof for the identity in part (b). [*Hint*: From a group of $n$ people, choose a committee of any size with one of the people in the committee designated as the chair. In how many ways can you do this?]

   (d) Let $[n] = \{1, 2, 3, \ldots, n\}$. Use the identity in part (b) to prove that the number of functions $f : \mathcal{P}([n]) \to \mathcal{P}([n])$ satisfying $f(X) \subseteq X$ for all $X \in \mathcal{P}([n])$ is exactly $2^{n2^{n-1}}$. [*Hint*: How many choices do you have for $f(\emptyset)$? How many for $f(\{1\})$?]

2. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer $n$.

   (a) Find all functions $f : [2] \to [2]$ such that $f(k) \leq k \ \forall k \in [2]$.

   (b) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (c) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (d) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k+1 \ \forall k \in [n]$.

   (e) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \leq k+1 \ \forall k \in [n]$.

3. Let $n \geq 3$ be an integer, and let $\{1, 2, \ldots, n\}$ be the vertices of the complete graph $K_n$.

   (a) Find two different circuits of length 3 (that is, 3 edges) in $K_4$ which use the same three vertices. [*Note*. According to the definition of circuit on page 667, two circuits are different if they are not exactly the same sequence of vertices and edges.]

   (b) Find the number of circuits of length 3 in $K_n$.

   (c) Find the number of subgraphs of $K_n$ which are connected and have exactly 3 vertices, all of degree 2.

   (d) Find the number of Hamiltonian circuits for $K_n$.

   (e) Find the number of subgraphs of $K_n$ which are connected and have $n$ vertices, all of degree 2.