MATH 271 ASSIGNMENT 4 SOLUTIONS

1. Let n be a positive integer.

(a) Prove that $\sum_{i=1}^{n+1} i\binom{n+1}{i} = \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=2}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}$. [*Hint*: Pascal's Formula (page 360).]

(b) Use part (a) and induction on n to prove the identity $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$ for all integers $n \geq 1$. [*Hint*: Replace $i-1$ by j in the last two sums in the formula in part (a). You may also use Example 6.7.2 on page 368.]

(c) Give a *combinatorial* proof for the identity in part (b). [*Hint*: From a group of n people, choose a committee of any size with one of the people in the committee designated as the chair. In how many ways can you do this?]

(d) Let $[n] = \{1, 2, 3, \ldots, n\}$. Use the identity in part (b) to prove that the number of functions $f : \mathcal{P}([n]) \to \mathcal{P}([n])$ satisfying $f(X) \subseteq X$ for all $X \in \mathcal{P}([n])$ is exactly $2^{n2^{n-1}}$. [*Hint*: How many choices do you have for $f(\emptyset)$? How many for $f(\{1\})$?]

(a) We get

$$\begin{aligned}
\sum_{i=1}^{n+1} i\binom{n+1}{i} &= \sum_{i=1}^{n+1} i\left[\binom{n}{i} + \binom{n}{i-1}\right] && \text{by Pascal's Formula (see (1) below)}\\
&= \sum_{i=1}^{n+1} i\binom{n}{i} + \sum_{i=1}^{n+1} i\binom{n}{i-1} && \text{by Theorem 4.1.1 part 1, page 207}\\
&= \sum_{i=1}^{n+1} i\binom{n}{i} + \sum_{i=1}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}\\
&= \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=2}^{n+1}(i-1)\binom{n}{i-1} + \sum_{i=1}^{n+1}\binom{n}{i-1}. && \text{(see (2) below)}
\end{aligned}$$

Notes. (1) According to page 360 of the text, Pascal's Formula $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ is only valid if $i \leq n$. Thus, strictly speaking, it does not apply in the case $i = n+1$ in line 1 above. However, when $i = n+1$ Pascal's Formula says $\binom{n+1}{n+1} = \binom{n}{n+1} + \binom{n}{n}$, where $\binom{n+1}{n+1} = \binom{n}{n} = 1$, so it would still be true if we defined $\binom{n}{n+1} = 0$. But since $\binom{n}{n+1}$ would mean the number of $(n+1)$-element subsets of an n-element set, it should be set equal to 0, in which case Pascal's Formula is okay for $i = n+1$ as well.

(2) Since we can define $\binom{n}{n+1} = 0$, in the last line above we changed the upper limit of the first right-hand sum from $n+1$ to n without changing the sum. Similarly we changed the lower limit of the second right-hand sum from $i = 1$ to $i = 2$, because when $i = 1$ the term in the sum is $0\binom{n}{0} = 0$.

(b) *Basis step*: When $n = 1$, the identity says that $1\binom{1}{1} = 1 \cdot 2^0$ or $1 = 1$ which is true.

Inductive step: Assume that

$$\sum_{i=1}^{k} i\binom{k}{i} = k2^{k-1}$$

is true for some integer $k \geq 1$. We want to prove that

$$\sum_{i=1}^{k+1} i\binom{k+1}{i} = (k+1)2^k.$$

Putting $i - 1 = j$ as in the Hint, and changing the limits of summation accordingly, we get

$$\begin{aligned}
\sum_{i=1}^{k+1} i\binom{k+1}{i} &= \sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=2}^{k+1}(i-1)\binom{k}{i-1} + \sum_{i=1}^{k+1}\binom{k}{i-1} && \text{by part (a)} \\
&= \sum_{i=1}^{k} i\binom{k}{i} + \sum_{j=1}^{k} j\binom{k}{j} + \sum_{j=0}^{k}\binom{k}{j} && \text{by the Hint} \\
&= k2^{k-1} + k2^{k-1} + \sum_{j=0}^{k}\binom{k}{j} && \text{by the assumption} \\
&= 2k2^{k-1} + 2^k && \text{by Example 6.7.2} \\
&= k2^k + 2^k = (k+1)2^k,
\end{aligned}$$

which finishes the inductive step.

Therefore, by induction, the identity is true for all integers $n \geq 1$.

(c) Let's choose the committee and chair by first choosing the people on the committee and then choosing the chair from among them. For each integer i between 1 and n, there are $\binom{n}{i}$ ways to choose a committee of i people from the n people, and then i ways to choose one of these i committee members to be the chair. So by the multiplication rule, there are $i\binom{n}{i}$ ways to choose a committee of i members including a chair. Any size i from 1 to n is possible, so by the addition rule, there must be $\sum_{i=1}^{n} i\binom{n}{i}$ ways to choose the committee and chair this way.

Next we choose the committee and chair by first choosing the chair, then the rest of the committee. There are n ways to choose one person (the chair) from among all n people. Whomever we choose, there are $n - 1$ people left who could be on the committee, and we can choose any subset of them to fill out the committee, so there are 2^{n-1} ways to choose the rest of the committee. By the multiplication rule, there are $n2^{n-1}$ ways to choose the committee and chair.

We have counted the same thing two different ways, so our answers must be equal. Thus the identity in part (b) must be true.

Note. For those of you who have taken calculus, here is another nice proof of the identity in part (b). By the Binomial Theorem,

$$\sum_{i=0}^{n}\binom{n}{i}x^i = (x+1)^n.$$

Take the derivative of both sides of this identity: we get

$$\sum_{i=0}^{n} \binom{n}{i} i x^{i-1} = n(x+1)^{n-1}.$$

Putting $x = 1$ (and removing $i = 0$ from the above sum) gives us the identity in part (b).

(d) To define one such function $f : \mathcal{P}([n]) \to \mathcal{P}([n])$, we need to say what $f(X)$ is for each $X \in \mathcal{P}([n])$.

- $f(\emptyset)$ must be a subset of $\emptyset$, so there is only one choice, namely $f(\emptyset) = \emptyset$.
- $f(\{1\})$ must be a subset of $\{1\}$, so there are two choices, $\emptyset$ and $\{1\}$.
- Similarly, for each of the one-element subsets $X = \{x\}$ of $[n]$, there will be only two choices for $f(X)$, because X will have only two subsets.
- In general, if X is a k-element subset of $[n]$, then X will have 2^k subsets, so there will be 2^k choices for $f(X)$.

For each k, there are $\binom{n}{k}$ subsets of $[n]$ of size k, so by the multiplication rule the number of ways to define $f(X)$ for all subsets X of $[n]$ of size k is $2^k \cdot 2^k \cdot \ldots \cdot 2^k$ ($\binom{n}{k}$ factors) which is $2^{k\binom{n}{k}}$. Thus by the multiplication rule the number of such functions f is

$$1 \cdot 2^{\binom{n}{1}} \cdot 2^{2\binom{n}{2}} \cdot \ldots \cdot 2^{n\binom{n}{n}} = 2^{\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}} = 2^{\sum_{i=1}^{n} i\binom{n}{i}} = 2^{n2^{n-1}} \qquad \text{by part (b).}$$

2. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer n.

(a) Find all functions $f : [2] \to [2]$ such that $f(k) \le k \ \forall k \in [2]$.

(b) Find the number of functions $f : [n] \to [n]$ such that $f(k) \le k \ \forall k \in [n]$.

(c) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \le k \ \forall k \in [n]$.

(d) Find the number of functions $f : [n] \to [n]$ such that $f(k) \le k + 1 \ \forall k \in [n]$.

(e) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \le k + 1 \ \forall k \in [n]$.

(a) For any such function f, since $f(1) \le 1$ and $f(2) \le 2$ we need $f(1) = 1$ and $f(2) = 1$ or 2. Thus there are exactly **two** such functions (which we call f_1 and f_2). They are respectively defined by:

- $f_1(1) = 1$ and $f_1(2) = 1$;
- $f_2(1) = 1$ and $f_2(2) = 2$.

(b) Since, for every k, $f(k)$ must be one of the k values $1, 2, \ldots, k$, there is one choice for $f(1)$ (namely 1), two choices for $f(2)$ (namely 1 or 2), and so on up to n choices for $f(n)$ (namely any of $1, 2, \ldots, n$). Thus by the multiplication rule there are $1 \cdot 2 \cdot \ldots \cdot n = n!$ ways to assign all the values $f(1), f(2), \ldots, f(n)$, that is, **n!** different functions.

(c) If f must be one-to-one, then we still must assign $f(1) = 1$, but then we cannot assign $f(2)$ to be 1 too, so we must put $f(2) = 2$. Next we cannot let $f(3)$ be 1 or 2, so we must put $f(3) = 3$. Continuing in this way, we are forced to put $f(k) = k$ for each k, so there is just **one** one-to-one function $f : [n] \to [n]$, namely the identity function.

(d) Proceeding as in part (b), for each k, $f(k)$ must be one of the $k+1$ choices $1, 2, \ldots, k+1$, provided that $k < n$. So $f(1)$ can be 1 or 2, $f(2)$ can be 1, 2 or 3, and so on up to $f(n-1)$ which can be any of $1, 2, \ldots, n$. But $f(n)$ must still belong to $[n]$ so there are only n choices for $f(n)$. Thus by the multiplication rule the total number of functions is $2 \cdot 3 \cdot \ldots \cdot n \cdot n = \mathbf{n(n!)}$.

(e) Note that since $[n]$ is finite, a function $f : [n] \to [n]$ is onto if and only if it is one-to-one. So we are really just counting one-to-one functions again. Now $f(1)$ must be 1 or 2, so there are two choices for $f(1)$. Then $f(2)$ must be 1, 2 or 3, so removing whichever choice we made for $f(1)$ will leave two choices for $f(2)$. In general there will be $k+1$ choices for $f(k)$ (namely $1, 2, \ldots, k+1$), but after we remove the choices we make for $f(1), f(2), \ldots, f(k-1)$ we will always have exactly two choices left for $f(k)$. The exception again is that for $f(n)$ there are only n choices originally (namely $1, 2, \ldots, n$), and after we remove the choices we make for $f(1), f(2), \ldots, f(n-1)$ we will only have one choice left for $f(n)$. So in total there will be $2 \cdot 2 \cdot \ldots \cdot 2 \cdot 1 = \mathbf{2^{n-1}}$ onto functions.

3. Let $n \geq 3$ be an integer, and let $\{1, 2, \ldots, n\}$ be the vertices of the complete graph K_n.

(a) Find two different circuits of length 3 (that is, 3 edges) in K_4 which use the same three vertices. [*Note.* According to the definition of circuit on page 667, two circuits are different if they are not exactly the same sequence of vertices and edges.]

(b) Find the number of circuits of length 3 in K_n.

(c) Find the number of subgraphs of K_n which are connected and have exactly 3 vertices, all of degree 2.

(d) Find the number of Hamiltonian circuits for K_n.

(e) Find the number of subgraphs of K_n which are connected and have n vertices, all of degree 2.

(a) For example, circuits 1231 and 1321 both use the three vertices $1, 2, 3$, but they are different circuits because the vertices are in a different order.

(b) Notice that a circuit of length 3 in K_n must use three different vertices, because K_n has no loops, and so we cannot repeat any vertices. Since listing the vertices is enough to determine the circuit, we only need to count the number of sequences a, b, c, a of vertices of K_n, where a, b, c are different. There are n choices for the first (and last) vertex a, and for each choice of a there are $n-1$ choices for the second vertex b and then $n-2$ choices for the third vertex c. By the multiplication rule, the number of such circuits is $\mathbf{n(n-1)(n-2)}$.

(c) If a subgraph of K_n is connected and has exactly 3 vertices, all of degree 2, it means that the subgraph must look exactly like a circuit of length 3. So once we choose the three vertices we know the subgraph. The difference between this part and part (b) is that in this part, it only matters what the vertices are, not what order they are in. Therefore the number of such subgraphs is $\binom{\mathbf{n}}{\mathbf{3}}$.

(d) Similar to part (b), we need to find the number of sequences $a_1, a_2, \ldots, a_n, a_1$, where $a_1, a_2, \ldots, a_n$ are just the vertices $1, 2, \ldots, n$ in some order. So the answer is $\mathbf{n!}$.

(e) Similar to part (c), if a subgraph of K_n is connected and has n vertices, all of degree 2, then it must be a Hamiltonian circuit of K_n. But two different Hamiltonian circuits will give us the same subgraph if they have the same edges. So we need to count how many Hamiltonian circuits of K_n will have the same edges, and divide the number of Hamiltonian circuits by this factor to eliminate the duplicate subgraphs. For each Hamiltonian circuit, we could start at any of the n vertices and go around the circuit in either direction, and we would get different circuits but the same subgraph. So we need to divide the answer to part (d) by $2n$, getting $n!/(2n) = \mathbf{(n-1)!/2}$ different subgraphs.