MATH 271 ASSIGNMENT 3 SOLUTIONS

1. Prove or disprove each of the following statements. Proofs should use the "element" methods given in Section 5.2. [*Note*: $\mathcal{P}(X)$ denotes the power set of the set X.]

   (a) For all sets A, B, C, $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

   (b) For all sets A, B, C, $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

   (c) For all sets A, B, C, $A \times (B - C) = (A \times B) - (A \times C)$.

   (d) For all sets A and B, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   (e) For all sets A and B, $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   (f) For all sets A and B, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

   (a) This inequality is **true**. Here is a proof.

   Let A, B, C be arbitrary sets. Note that the left side of this inequality is a Cartesian product, which means that its elements will be ordered pairs. So let (a, b) be an arbitrary element of $A \times (B - C)$. This means that $a \in A$ and $b \in B - C$. Since $b \in B - C$, this means that $b \in B$ and $b \notin C$. Since $a \in A$ and $b \in B$, we get that $(a, b) \in A \times B$. But since $b \notin C$, we know that (a, b) cannot be an element of $A \times C$. Since $(a, b) \in A \times B$ but $(a, b) \notin A \times C$, we know $(a, b) \in (A \times B) - (A \times C)$. Therefore $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

   (b) Similarly, this inequality is **true**, and we can reverse our steps in part (a) to get a proof.

   Let (a, b) be an arbitrary element of $(A \times B) - (A \times C)$. This means that $(a, b) \in A \times B$ but $(a, b) \notin A \times C$. Since $(a, b) \in A \times B$, we know that $a \in A$ and $b \in B$. But since $(a, b) \notin A \times C$ although $a \in A$, we also know $b \notin C$. Thus $b \in B$ and $b \notin C$, which means $b \in B - C$. Thus $(a, b) \in A \times (B - C)$. Therefore $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

   (c) Since the inequalities in parts (a) and (b) both hold, we get that the equality in (c) holds for all sets A, B, C.

   (d) This inequality is **false**, and counterexamples are not hard to find. For example, let $A = \{1, 2\}$ and $B = \{1\}$. Then $\{1, 2\} \subseteq A$ and $\{1, 2\} \not\subseteq B$, so $\{1, 2\} \in \mathcal{P}(A)$ and $\{1, 2\} \notin \mathcal{P}(B)$, so $\{1, 2\} \in \mathcal{P}(A) - \mathcal{P}(B)$. However $A - B = \{2\}$, so $\{1, 2\} \notin \mathcal{P}(A - B)$. Therefore $\mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B)$.

   (e) This inequality is **false** no matter what sets we choose for A and B! To see this, let A and B be any sets. Notice that the empty set $\emptyset \subseteq A - B$ regardless of what A and B are, so $\emptyset \in \mathcal{P}(A - B)$. However, since $\emptyset \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(B)$, we get $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$. Therefore $\mathcal{P}(A - B) \not\subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   Note. You can prove that if X is any *nonempty* set so that $X \in \mathcal{P}(A - B)$, then $X \in \mathcal{P}(A) - \mathcal{P}(B)$. So the only counterexample to the inequality in part (e) is the empty set.

   (f) Since the inequality in (d) (or (e)) fails, the equality in (f) fails too.

2. (a) Prove that

$$n - \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil \quad \text{for all positive integers } n.$$

Here $\lceil x \rceil$ denotes the *ceiling* of the number x, as defined in §3.5. [*Hint*: do the cases n odd and n even separately.]

(b) The sequence $A_0, A_1, A_2, \ldots$ of sets is defined as follows:

$$A_0 = \emptyset, \quad \text{and } A_n = \{1, 2, \ldots, n\} - A_{n-1} \text{ for all integers } n \geq 1.$$

Find A_1, A_2 and A_3.

(c) For the sets A_n defined in part (b), prove **by induction on** n that $N(A_n) = \lceil n/2 \rceil$ for every integer $n \geq 0$. [*Hint*: Theorem 6.3.2 on page 322. $N(X)$ denotes the number of elements in the set X.]

(a) *Case* 1: Assume n is odd. Then $n = 2k + 1$ for some integer k. So

$$\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{2k}{2} \right\rceil = k$$

and

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = \left\lceil k + \frac{1}{2} \right\rceil = k + 1,$$

so

$$n - \left\lceil \frac{n-1}{2} \right\rceil = (2k+1) - k = k + 1 = \left\lceil \frac{n}{2} \right\rceil.$$

Case 2: Assume n is even. Then $n = 2k$ for some integer k. So

$$\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{2k-1}{2} \right\rceil = \left\lceil k - \frac{1}{2} \right\rceil = k$$

and

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k}{2} \right\rceil = k,$$

so

$$n - \left\lceil \frac{n-1}{2} \right\rceil = 2k - k = k = \left\lceil \frac{n}{2} \right\rceil.$$

(b) We get

$$A_1 = \{1\} - A_0 = \{1\} - \emptyset = \{1\},$$
$$A_2 = \{1, 2\} - A_1 = \{1, 2\} - \{1\} = \{2\},$$
$$A_3 = \{1, 2, 3\} - A_2 = \{1, 2, 3\} - \{2\} = \{1, 3\}.$$

(c) *Basis step.* When $n = 0$, $\lceil n/2 \rceil = 0 = N(A_0)$ since $A_0 = \emptyset$.

Inductive step. Assume that $N(A_k) = \lceil k/2 \rceil$ for some integer $k \geq 0$. We want to prove that $N(A_{k+1}) = \lceil (k+1)/2 \rceil$. Note that $A_0 \subseteq \{1\}$ and (for $k > 0$)

$$A_k = \{1, 2, \ldots, k\} - A_{k-1} \subseteq \{1, 2, \ldots, k+1\}.$$

Thus

$$\begin{aligned} N(A_{k+1}) &= N(\{1, 2, \ldots, k+1\} - A_k) \quad \text{by recursion} \\ &= N(\{1, 2, \ldots, k+1\}) - N(A_k) \quad \text{by Theorem 6.3.2 on page 322} \\ &= (k+1) - \left\lceil \frac{k}{2} \right\rceil \quad \text{by assumption} \\ &= \left\lceil \frac{k+1}{2} \right\rceil \quad \text{by part (a) (using } n = k+1\text{),} \end{aligned}$$

which is what we wanted to prove.

So by induction, $N(A_n) = \lceil n/2 \rceil$ for every integer $n \geq 0$.

3. A licence plate consists of any three letters (from the usual 26-letter alphabet) followed by any three digits. Find the number of licence plates with each of the following properties. You need not simplify your answers.

   (a) They contain exactly two different symbols.

   (b) They contain exactly three different symbols.

   (c) They contain at least two 8's, but no 4, and the letters HAL in some order.

   (d) They use three different letters in alphabetical order and three different digits in increasing order. [*Hint*: start by choosing the three letters.]

(a) For a licence plate to have exactly two different symbols, it must have only one kind of letter and only one kind of digit; for example AAA111 is such a licence plate. The number of choices for the letter is 26 and the number of choices for the digit is 10. So by the multiplication rule, the number of such licence plates is $26 \times 10 =$ **260**.

(b) For a licence plate to have exactly three different symbols, it must have either (i) one kind of letter and two kinds of digits (for example AAA112), or (ii) two kinds of letters and one kind of digit (for example AAB111). We count these two possibilities separately.

(i) The number of choices for the letter is 26. For the two digits, one of them (say x) will occur twice and the other (say y) only once. The number of ways to choose x is 10, and (no matter which digit we choose for x) the number of ways to then choose y is 9. There are three places for the single digit y to go, and then the two x's will have to go in the other two places reserved for the digits. So by the multiplication rule, the number of licence plates of type (i) is $26 \times 10 \times 9 \times 3 =$ **7020**.

(ii) We similarly count these licence plates. The number of choices for the digit is 10. For the two letters, one of them (say α) will occur twice and the other (say β) only once. The number of ways to choose α is 26, and (no matter which letter we choose for α) the number of ways to then choose β is 25. There are three places for the single letter β to go, and then the two α's will have to go in the other two places reserved for the letters. So by the multiplication rule, the number of licence plates of type (ii) is $10 \times 26 \times 25 \times 3 = \mathbf{19500}$.

Thus, by the addition rule, the total number of such licence plates is

$$7020 + 19500 = \mathbf{26520}.$$

(c) The number of ways to arrange the letters HAL in some order is $3! = 6$. If our third digit (besides the two 8's) is another 8, then there is only one way to arrange these three 8's, so the total number of such licence plates will be just **6**. On the other hand, if our third digit is not an 8, then there are 8 choices for it (any digit except 8 and 4). There are three places to put this third digit, so there are $6 \times 8 \times 3 = \mathbf{144}$ such licence plates this time. So by the addition rule, the total number of licence plates will be $6 + 144 = \mathbf{150}$.

(d) There are $\binom{26}{3}$ ways to choose three different letters, and only one way to arrange them, since they must be in alphabetical order. There are $\binom{10}{3}$ ways to choose three different digits, and only one way to arrange them, since they must be in increasing order. By the multiplication rule, the number of such licence plates is

$$\binom{26}{3}\binom{10}{3} = \frac{26 \times 25 \times 24}{3 \times 2} \cdot \frac{10 \times 9 \times 8}{3 \times 2} = 26 \times 25 \times 4 \times 10 \times 3 \times 4 = \mathbf{312000}.$$