MATH 271 ASSIGNMENT 2 SOLUTIONS

1. Let

$$S_n = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(2n-1)(2n)} \ .$$

(a) Find and simplify S_1, S_2 and S_3.

(b) Use part (a) (and more data if you need it) to guess a simple formula for S_n for any positive integer n.

(c) **Use mathematical induction** (or well ordering) to prove that your guess in part (b) is true for all positive integers n.

(a) We get

- $S_1 = \dfrac{1}{1\cdot 2} = \dfrac{1}{2}$,
- $S_2 = \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} = \dfrac{1}{6} + \dfrac{1}{12} = \dfrac{3}{12} = \dfrac{1}{4}$,
- $S_3 = \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \dfrac{1}{5\cdot 6} = \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} = \dfrac{5+3+2}{60} = \dfrac{10}{60} = \dfrac{1}{6}$.

(b) From part (a) we might guess that

$$S_n = \frac{1}{2n} \quad \text{for all integers } n \geq 1.$$

(c) *Basis step.* $S_n = 1/(2n)$ is true for $n = 1$ (and also for $n = 2$ and $n = 3$), by part (a).

Inductive step. Assume that $S_k = 1/(2k)$ for some integer $k \geq 1$. This means that

$$\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(2k-1)(2k)} = \frac{1}{2k} \ .$$

We want to prove that $S_{k+1} = 1/(2(k+1))$, which means we want to prove that

$$\frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots + \frac{1}{(2k+1)(2k+2)} = \frac{1}{2(k+1)} \ .$$

We notice that

$$\begin{aligned}
&\frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots + \frac{1}{(2k+1)(2k+2)} \\
&= \left[\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(2k-1)(2k)}\right] + \frac{1}{(2k)(2k+1)} \\
&\qquad + \frac{1}{(2k+1)(2k+2)} - \frac{1}{k(k+1)} \\
&= \frac{1}{2k} + \frac{1}{(2k)(2k+1)} + \frac{1}{(2k+1)(2k+2)} - \frac{1}{k(k+1)} \qquad \text{by assumption} \\
&= \frac{(2k+1)(k+1) + (k+1) + k - 2(2k+1)}{2k(2k+1)(k+1)} \\
&= \frac{2k^2 + 3k + 1 + k + 1 + k - 4k - 2}{2k(2k+1)(k+1)} \\
&= \frac{2k^2 + k}{2k(2k+1)(k+1)} = \frac{1}{2(k+1)} \ ,
\end{aligned}$$

so the statement is true for $n = k + 1$. Therefore by induction, the statement is true for all integers $n \geq 1$.

Note. There is a nice short proof (not using induction) that $S_n = \frac{1}{2n}$ for all integers $n \geq 1$, using *telescoping*, which is a special technique mentioned in Example 4.1.10 on page 205 of the text. It makes use of the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for all $k > 0$. Then

$$\begin{aligned} S_n &= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(2n-1)(2n)} \\ &= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}\,. \qquad \text{(everything else cancels out)} \end{aligned}$$

2. The sequence $b_0, b_1, b_2, \ldots$ is defined by: $b_0 = 1$, $b_1 = 2$ and $b_n = 3b_{n-1} + b_{n-2}$ for all integers $n \geq 2$.

   (a) Calculate b_2, b_3 and b_4.

   (b) **Use mathematical induction** (or well ordering) to prove that $\gcd(b_{n+1}, b_n) = 1$ for all integers $n \geq 0$. [You may use Lemma 3.8.2 on page 193.]

   (c) **Use strong induction** (or well ordering) to prove that $b_n \leq 4^n$ for all integers $n \geq 0$.

   (a) We get

   $$b_2 = 3b_1 + b_0 = 3\cdot 2 + 1 = \mathbf{7}, \quad b_3 = 3b_2 + b_1 = 3\cdot 7 + 2 = \mathbf{23}, \quad b_4 = 3b_3 + b_2 = 3\cdot 23 + 7 = \mathbf{76}.$$

   (b) *Basis step.* When $n = 0$ we have

   $$\gcd(b_1, b_0) = \gcd(2, 1) = 1,$$

   so the statement is true for $n = 0$.

   Inductive step. Assume that $\gcd(b_{k+1}, b_k) = 1$ for some integer $k \geq 0$. We want to prove that $\gcd(b_{k+2}, b_{k+1}) = 1$. Since $b_{k+2} = 3b_{k+1} + b_k$, we know that

   $$\begin{aligned} \gcd(b_{k+2}, b_{k+1}) &= \gcd(b_{k+1}, b_k) \qquad \text{from Lemma 3.8.2} \\ &= 1 \qquad \text{by assumption,} \end{aligned}$$

   so the statement is true for $n = k + 1$. Therefore by induction, $\gcd(b_{n+1}, b_n) = 1$ for all integers $n \geq 0$.

   (c) *Basis step.* When $n = 0$ the statement says $b_0 = 1 \leq 4^0$, which is true since $4^0 = 1$. When $n = 1$ the statement says $b_1 = 2 \leq 4^1$, which is also true since $4^1 = 4$.

   Inductive step. Assume that $b_i \leq 4^i$ for all integers i satisfying $0 \leq i < k$, for some integer $k \geq 2$. We want to prove that $b_k \leq 4^k$. Since $k \geq 2$, we know by assumption that $b_{k-1} \leq 4^{k-1}$ and $b_{k-2} \leq 4^{k-2}$. Thus

   $$\begin{aligned} b_k &= 3b_{k-1} + b_{k-2} \\ &\leq 3 \cdot 4^{k-1} + 4^{k-2} \qquad \text{by assumption} \\ &= 4^{k-2}(3 \cdot 4 + 1) = 4^{k-2} \cdot 13 < 4^{k-2} \cdot 4^2 = 4^k, \end{aligned}$$

   so the statement is true for $n = k$. Therefore $b_n \leq 4^n$ for all integers $n \geq 0$, by strong induction.

3. You are given the following "while" loop:

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[Pre-condition: $m$ is a nonnegative integer, $a = 0$, $b = 1$, $i = 0$.]
while ($i \neq m$)
        1. $a := 2b - a$
        2. $b := 3a - 2b$
        3. $i := i + 1$
end while
[Post-condition: $b - a = 2^m$.]
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Loop invariant $I(n)$ is: $i = n, \quad a = 2^{n+1} - 2, \quad b = 2^{n+1} + 2^n - 2.$

(a) Prove the correctness of this loop with respect to the pre- and post-conditions.

(b) Suppose the "while" loop is as above, with the same pre-condition, except that statement 2 is replaced by: $b := 3a - 2b - 1$. Run through this new loop a few times to get data. Then find a post-condition that gives the final value of $b - a$, and an appropriate loop invariant, and prove the correctness of this new loop.

(a) We first need to check that the loop invariant holds when $n = 0$. But $I(0)$ says $i = 0$, $a = 2^1 - 2 = 0$, and $b = 2^1 + 2^0 - 2 = 1$, and these are all true by the pre-conditions.

So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$ where $k < m$. We want to prove that $I(k+1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that

$$i = k, \quad a = 2^{k+1} - 2, \quad b = 2^{k+1} + 2^k - 2,$$

and we now go through the loop.

- Step 1:

$$\begin{aligned} a := 2b - a &= 2(2^{k+1} + 2^k - 2) - (2^{k+1} - 2) \\ &= 2^{k+2} + 2^{k+1} - 4 - 2^{k+1} + 2 \\ &= 2^{k+2} - 2, \end{aligned}$$

  which agrees with the formula for a in $I(k+1)$.

- Step 2:

$$\begin{aligned} b := 3a - 2b &= 3(2^{k+2} - 2) - 2(2^{k+1} + 2^k - 2) \\ &= 3 \cdot 2^{k+2} - 6 - 2^{k+2} - 2^{k+1} + 4 \\ &= 2^{k+2} + (2^{k+2} - 2^{k+1}) - 2 \\ &= 2^{k+2} + 2^{k+1} - 2, \end{aligned}$$

  which agrees with the formula for b in $I(k+1)$.

- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k+1)$.

Thus $I(k+1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $a = 2^{m+1} - 2$ and $b = 2^{m+1} + 2^m - 2$, and so $b - a = 2^m$ as required in the post-condition.

(b) If we set the variables to their pre-condition values of $a = 0$, $b = 1$ and $i = 0$, and run through the loop, the new values we get are

$$a = 2 \cdot 1 - 0 = 2, \quad b = 3 \cdot 2 - 2 \cdot 1 - 1 = 3, \quad i = 1.$$

If we continue to run through the loop, and keep track of the variables in a table, here is what we get:

n	0	1	2	3	4	5	6
a	0	2	4	6	8	10	12
b	1	3	5	7	9	11	13
i	0	1	2	3	4	5	6

It certainly looks like the post-condition should be $b - a = 1$ no matter what m is, and the loop invariant $I(n)$ should be: $i = n, \quad a = 2n, \quad b = 2n + 1$. From the pre-condition, $I(0)$ is true. So assume that $I(k)$ holds for some integer $k \geq 0$ where $k < m$, and we want to prove that $I(k + 1)$ holds. So we are assuming that

$$i = k, \quad a = 2k, \quad b = 2k + 1,$$

and we now go through the loop.

- Step 1: $a := 2b - a = 2(2k + 1) - 2k = 2k + 2 = 2(k + 1)$,
  which agrees with the formula for a in $I(k + 1)$.
- Step 2: $b := 3a - 2b - 1 = 3(2k + 2) - 2(2k + 1) - 1 = 2k + 3 = 2(k + 1) + 1$,
  which agrees with the formula for b in $I(k + 1)$.
- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k + 1)$.

Thus $I(k + 1)$ is true, as required.

Finally the loop stops when $i = m$, and then the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $a = 2m$ and $b = 2m + 1$, and so $b - a = 1$ as required in the post-condition.