MATH 271 ASSIGNMENT 1 SOLUTIONS

1. For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation. You may use that every integer is either even or odd (but not both), but otherwise use only the definitions of even and odd integers.

   (a) $\forall a, b \in \mathbb{Z}$, if a is even and $a|b$ then b is even.

   (b) $\forall a, b \in \mathbb{Z}$, if a is even and $b|a$ then b is even.

   (c) $\forall a, b \in \mathbb{Z}$, if a is odd and $a|b$ then b is odd.

   (d) $\forall a, b \in \mathbb{Z}$, if a is odd and $b|a$ then b is odd.

   (a) This statement is **true**. Here is a proof.

   Let $a, b \in \mathbb{Z}$ so that a is even and $a|b$. This means that $a = 2k$ and $b = a\ell$ for some $k, \ell \in \mathbb{Z}$. Thus $b = a\ell = 2(k\ell)$ where $k\ell$ is an integer. So b is even by definition.

   (b) This statement is **false**. A counterexample is $a = 2$, $b = 1$. Then $a = 2$ is even and $b|a$ (since $1|2$), but $b = 1$ is not even.

   (c) This statement is **false**. A counterexample is $a = 1$, $b = 2$. Then $a = 1$ is odd and $a|b$ (since $1|2$), but $b = 2$ is not odd.

   (d) This statement is **true**. Here is a proof.

   Let $a, b \in \mathbb{Z}$ so that a is odd and $b|a$. We want to prove that b is odd. We will do this by contradiction. Suppose that b is not odd, which means we suppose that b is even. Then, since we have b even and $b|a$, we know from part (a) that a must be even. But this contradicts the assumption that a is odd. Thus by contradiction, b must be odd.

2. Prove or disprove each of the following:

   (a) $\forall a \in \mathbb{N}\ \exists b \in \mathbb{N}$ so that ab is composite.

   (b) $\exists a \in \mathbb{N}$ so that $\forall b \in \mathbb{N}$, ab is composite.

   (c) $\forall a \in \mathbb{N}\ \exists b \in \mathbb{N}$ so that $a + b$ is composite.

   (d) $\exists a \in \mathbb{N}$ so that $\forall b \in \mathbb{N}$, $a + b$ is composite.

   (a) This statement is **true**. Here is a proof.

   Let $a \in \mathbb{N}$ be arbitrary. We choose $b = 4$ regardless of the value of a. Then $ab = 4a = 2 \cdot 2a$, where both 2 and $2a$ are integers greater than 1. Therefore ab is composite.

   (b) This statement is **true**. Here is a proof.

   Choose $a = 4$. Then for any $b \in \mathbb{N}$, $ab = 4b = 2 \cdot 2b$, where both 2 and $2b$ are integers greater than 1. Therefore ab is composite.

   (c) This statement is **true**. Here is a proof.

   Let $a \in \mathbb{N}$ be arbitrary. We choose $b = 3a$ which is a positive integer. Then $a+b = 4a = 2 \cdot 2a$, where both 2 and $2a$ are integers greater than 1. Therefore $a + b$ is composite.

(d) This statement is **false**. We prove this by proving that its negation is true.

The negation of this statement is:

$$\forall a \in \mathbb{N}\ \exists b \in \mathbb{N} \text{ so that } a+b \text{ is not composite.}$$

Since $a+b \geq 2$ for all $a, b \in \mathbb{N}$, we can rewrite the negation as:

$$\forall a \in \mathbb{N}\ \exists b \in \mathbb{N} \text{ so that } a+b \text{ is prime.}$$

We want to prove this. Let $a \in \mathbb{N}$ be arbitrary. Since there are infinitely many primes (Theorem 3.7.4 on page 183), there must exist some prime $p > a$. Let $b = p - a$ which is a positive integer. Then $a + b = p$ which is prime, so the negation is true. Therefore the original statement is false.

3. For this question, do not use Exercises 13–16 on page 146 without proof.

(a) Prove or disprove: $\forall x \in \mathbb{Q}$, if $x \in \mathbb{Z}$ then $x\lfloor x \rfloor \in \mathbb{Z}$.

(b) Write out the converse of the statement in (a). Is it true? Give a proof or disproof.

(c) Prove or disprove: $\forall x \in \mathbb{R}$, if $x \in \mathbb{Q}$ then $x\lfloor x \rfloor \in \mathbb{Q}$.

(d) Write out the converse of the statement in (c). Is it true? Give a proof or disproof.

(a) This statement is **true**. Here is a proof.

Let $x \in \mathbb{Z}$ be arbitrary. Then we know that $\lfloor x \rfloor = x$, so $x\lfloor x \rfloor = x^2 \in \mathbb{Z}$.

(b) The converse is: $\forall x \in \mathbb{Q}$, if $x\lfloor x \rfloor \in \mathbb{Z}$ then $x \in \mathbb{Z}$.

The converse is **false**. A counterexample is $x = 5/2$ which is a rational number. Then $x\lfloor x \rfloor = (5/2) \cdot \lfloor 5/2 \rfloor = (5/2) \cdot 2 = 5$ which is an integer, but $x = 5/2$ is not an integer.

(c) This statement is **true**. Here is a proof.

Let $x \in \mathbb{Q}$ be arbitrary. This means that $x = a/b$ for some integers a, b where $b \neq 0$. Also, $\lfloor x \rfloor \in \mathbb{Z}$. Thus $x\lfloor x \rfloor = (a/b)\lfloor x \rfloor = (a\lfloor x \rfloor)/b$ where $a\lfloor x \rfloor$ is an integer. Therefore $x\lfloor x \rfloor \in \mathbb{Q}$ by definition.

(d) The converse is: $\forall x \in \mathbb{R}$, if $x\lfloor x \rfloor \in \mathbb{Q}$ then $x \in \mathbb{Q}$.

The converse is **false**. A counterexample is $x = \sqrt{2}/2$. Since $0 < \sqrt{2}/2 < 1$, we get $\lfloor x \rfloor = \lfloor \sqrt{2}/2 \rfloor = 0$ and so $x\lfloor x \rfloor = 0$ which is a rational number. However we claim that $x = \sqrt{2}/2$ is not a rational number. To prove this we can use contradiction. Suppose that $\sqrt{2}/2$ is rational, which means that $\sqrt{2}/2 = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $\sqrt{2} = (2a)/b$ where $2a \in \mathbb{Z}$, so $\sqrt{2}$ would be rational, which is a contradiction (because of Theorem 3.7.1 on page 181).