MATH 271 ASSIGNMENT 1 SOLUTIONS

1. For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.

   (a) $\forall x \in \mathbb{Q}$, x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a$.

   (b) $\exists x \in \mathbb{Q}$ so that x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a$.

   (c) $\forall x \in \mathbb{Q}$, x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a^2$.

   (d) $\forall x \in \mathbb{R}$, if x is irrational then x^2 is irrational or x^3 is irrational.

   (e) $\forall x \in \mathbb{R}$, if x^2 is rational then x^3 is rational or x^5 is rational.

   (a) This statement is **false**. The negation of the statement is

   $$\exists x \in \mathbb{Q} \text{ so that } x \neq a/b \text{ for any } a, b \in \mathbb{Z} \text{ such that } b|a.$$

   An example which proves the negation is $x = 1/2$, because if $x = a/b$ and $b|a$, it would mean that x is an integer, and $1/2$ is not an integer.

   (b) This statement is **true**. An example proving this statement is $x = 1$, which can be written as $1/1$ where $1|1$.

   (c) This statement is **true**. Here is a proof. Let $x \in \mathbb{Q}$. This means that $x = c/d$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Then $x = \frac{cd}{d^2}$ where $d^2 \neq 0$, and $(cd)^2 = c^2d^2$ where $c^2 \in \mathbb{Z}$, so $d^2|(cd)^2$. Thus we can put $a = cd$ and $b = d^2$ and we get $x = a/b$ where $a, b \in \mathbb{Z}$ and $b|a^2$. Done.

   (d) This statement is **true**. We prove it by proving the *contrapositive* instead. The contrapositive is

   $$\forall x \in \mathbb{R}, \text{ if } x^2 \text{ is rational and } x^3 \text{ is rational, then } x \text{ is rational.}$$

   Assume that $x \in \mathbb{R}$ is such that x^2 is rational and x^3 is rational. We want to prove that x is rational. There are two cases.

   Case 1. Suppose $x = 0$. Then of course x is rational, so we are done.

   Case 2. Now suppose that $x \neq 0$. Since x^2 is rational, $x^2 = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and also $a \neq 0$ (since $x \neq 0$). Since x^3 is rational, $x^3 = c/d$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Thus $x = x^3/x^2 = (c/d)/(a/b) = bc/ad$, where bc and ad are integers with $ad \neq 0$ (since a and d are each nonzero), so x is rational.

   (e) This statement is **false**. The negation is

   $$\exists x \in \mathbb{R} \text{ so that } x^2 \text{ is rational but } x^3 \text{ is irrational and } x^5 \text{ is irrational.}$$

   This statement is proved by the example $x = \sqrt{2}$. Then $x^2 = 2$ is rational, but $x^3 = 2\sqrt{2}$ and $x^5 = 4\sqrt{2}$ are both irrational. Both of these statements follow from Exercise 10 on page 178 of the text, or could be proved by contradiction. For example, to prove that $2\sqrt{2}$ is irrational, we assume that it is rational.This means that $2\sqrt{2} = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. But then $\sqrt{2} = a/(2b)$ where $2b \in \mathbb{Z}$ and $2b \neq 0$, so $\sqrt{2}$ is rational, which is impossible. Therefore $2\sqrt{2}$ must be irrational. Similarly $4\sqrt{2}$ is irrational.

2. (a) Disprove the following statement: $\forall x, y \in \mathbb{R}$, if $\lfloor xy \rfloor = 0$ then $\lfloor x \rfloor = 0$ or $\lfloor y \rfloor = 0$.
   (b) Write out the contrapositive of the statement in part (a). Is it true or false? Explain.
   (c) Write out the converse of the statement in part (a). Is it true or false? Explain.
   (d) Prove or disprove the following statement: $\forall x, y \in \mathbb{R}$, if $\lceil xy \rceil = 0$ then $\lceil x \rceil = 0$ or $\lceil y \rceil = 0$.
   (e) Prove or disprove the following statement: $\exists x, y \in \mathbb{R}$ such that $\lfloor xy \rfloor = 0$ and $\lfloor x \rfloor \lfloor y \rfloor = 271$.
   (f) Prove or disprove the following statement: $\exists x, y \in \mathbb{R}$ such that $\lceil xy \rceil = 0$ and $\lceil x \rceil \lceil y \rceil = 271$.

   (a) To disprove this statement, we need to find real numbers x and y so that $\lfloor xy \rfloor = 0$ but $\lfloor x \rfloor \neq 0$ and $\lfloor y \rfloor \neq 0$. An example is $x = -1/2$ and $y = -1$. Then $xy = 1/2$, so $\lfloor xy \rfloor = \lfloor 1/2 \rfloor = 0$ while $\lfloor x \rfloor = \lfloor -1/2 \rfloor = -1 \neq 0$ and $\lfloor y \rfloor = \lfloor -1 \rfloor = -1 \neq 0$.

   Note: if x and y are both positive numbers so that $\lfloor xy \rfloor = 0$, then $0 < xy < 1$, so either $0 < x < 1$ or $0 < y < 1$, so one of $\lfloor x \rfloor = 0$ or $\lfloor y \rfloor = 0$ will have to be true. So for a counterexample we have to look at negative x and y.

   (b) The contrapositive is

$$\forall x, y \in \mathbb{R}, \text{ if } \lfloor x \rfloor \neq 0 \text{ and } \lfloor y \rfloor \neq 0 \text{ then } \lfloor xy \rfloor \neq 0.$$

   The contrapositive is **false** because it is equivalent to the original statement which is false.

   (c) The converse is

$$\forall x, y \in \mathbb{R}, \text{ if } \lfloor x \rfloor = 0 \text{ or } \lfloor y \rfloor = 0 \text{ then } \lfloor xy \rfloor = 0.$$

   The converse is **false**. A counterexample is $x = 1/2$ and $y = 2$. Then $\lfloor x \rfloor = \lfloor 1/2 \rfloor = 0$, so the "if" part is true, but $xy = 1$, so $\lfloor xy \rfloor = \lfloor 1 \rfloor = 1 \neq 0$, so the "then" part is false.

   (d) This statement is **false**. A counterexample is $x = 1/2$ and $y = -1$. Then $xy = -1/2$, so $\lceil xy \rceil = \lceil -1/2 \rceil = 0$, but $\lceil x \rceil = \lceil 1/2 \rceil = 1 \neq 0$ and $\lceil y \rceil = \lceil -1 \rceil = -1 \neq 0$.

   (e) This statement is **true**. An example is $x = -271$ and $y = -1/272$. Then $xy = 271/272$, so $\lfloor xy \rfloor = \lfloor 271/272 \rfloor = 0$, and $\lfloor x \rfloor \lfloor y \rfloor = \lfloor -271 \rfloor \lfloor -1/272 \rfloor = (-271)(-1) = 271$.

   (f) This statement is **false**. We prove this by contradiction. Suppose that the statement were true. Let x and y be real numbers so that $\lceil xy \rceil = 0$ and $\lceil x \rceil \lceil y \rceil = 271$. Since $\lceil xy \rceil = 0$, we must have $-1 < xy \leq 0$, so xy is negative (or zero). This means that (without loss of generality) $x \leq 0$ and $y \geq 0$. But then $\lceil x \rceil \leq 0$ and $\lceil y \rceil \geq 0$, and thus $\lceil x \rceil \lceil y \rceil \leq 0$, which contradicts $\lceil x \rceil \lceil y \rceil = 271$.

3. Let N be your student ID number.

   (a) **Use the Euclidean Algorithm** to find $\gcd(N, 271)$.
   (b) Use your answer to part (a) to write $\gcd(N, 271)$ in the form $Na + 271b$ where $a, b \in \mathbb{Z}$.
   (c) Suppose that M is a positive integer such that $\gcd(M, 271) = \gcd(M, 2008)$. Find $\gcd(M, 271)$. Explain. [*Hint*: 271 is prime.]

(d) Suppose that M is an integer between 250000 and 450000 such that $\gcd(M, 271) = \gcd(M, 2008) + 20$. Find M. Explain. [You may use Exercise 33, page 631.]

(a) Let's do it for the hypothetical student number $N = 123456$. The Euclidean algorithm gives:

$$\begin{aligned} 123456 &= 455 \cdot 271 + 151 & &(\text{so } 151 = 123456 - 455 \cdot 271) \\ 271 &= 1 \cdot 151 + 120 & &(\text{so } 120 = 271 - 151) \\ 151 &= 1 \cdot 120 + 31 & &(\text{so } 31 = 151 - 120) \\ 120 &= 3 \cdot 31 + 27 & &(\text{so } 27 = 120 - 3 \cdot 31) \\ 31 &= 1 \cdot 27 + 4 & &(\text{so } 4 = 31 - 27) \\ 27 &= 6 \cdot 4 + 3 & &(\text{so } 3 = 27 - 6 \cdot 4) \\ 4 &= 1 \cdot 3 + 1 & &(\text{so } 1 = 4 - 3) \\ 3 &= 3 \cdot 1, \end{aligned}$$

so $\gcd(123456, 271) = \mathbf{1}$, the last nonzero remainder.

(b) Now, starting with the second-last equation above, solving it for the gcd 1, and plugging in the remainders one by one from the earlier equations, we get:

$$\begin{aligned} 1 &= 4 - 3 \\ &= 4 - (27 - 6 \cdot 4) = 7 \cdot 4 - 27 \\ &= 7 \cdot (31 - 27) - 27 = 7 \cdot 31 - 8 \cdot 27 \\ &= 7 \cdot 31 - 8 \cdot (120 - 3 \cdot 31) = 7 \cdot 31 - 8 \cdot 120 + 24 \cdot 31 = 31 \cdot 31 - 8 \cdot 120 \\ &= 31 \cdot (151 - 120) - 8 \cdot 120 = 31 \cdot 151 - 39 \cdot 120 \\ &= 31 \cdot 151 - 39 \cdot (271 - 151) = 70 \cdot 151 - 39 \cdot 271 \\ &= 70 \cdot (123456 - 455 \cdot 271) - 39 \cdot 271 = 70 \cdot 123456 - 31850 \cdot 271 - 39 \cdot 271 \\ &\qquad = 70 \cdot 123456 - 31889 \cdot 271. \end{aligned}$$

So $a = 70$ and $b = -31889$ in this case.

(c) Since $\gcd(M, 271)$ is a divisor of 271 and 271 is prime, $\gcd(M, 271)$ must be either 1 or 271. Since 271 does not divide into 2008, $\gcd(M, 2008)$ cannot be 271. Thus, since $\gcd(M, 271) = \gcd(M, 2008)$, $\gcd(M, 271) = 1$.

(d) Since $\gcd(M, 271) = 1$ or 271 and $\gcd(M, 271) = \gcd(M, 2008) + 20 > 20$, $\gcd(M, 271)$ must equal 271 and thus $\gcd(M, 2008)$ must equal 251. Thus M must be a multiple of both 271 and 251, and so, since $\gcd(271, 251) = 1$ since 271 is prime, by Exercise 33, page 631, $271 \cdot 251 = 68021$ must divide into M. Thus $M = 68021k$ for some positive integer k. Moreover, since $\gcd(M, 2008) = 251$ is odd and 2008 is even, M cannot be even. So k must be odd. Now $68021 \cdot 3 = 204063$ is too small to be M and $68021 \cdot 7 = 476147$ is too large to be M, so M must be $68021 \cdot 5 = \mathbf{340105}$ which lies in the right range for M.

Note. By the way, $\gcd(M, 2008) = 251$ is possible because $251 | 2008$. In fact, $2008 = 251 \cdot 8$ so $\gcd(M, 2008) = \gcd(340105, 2008) = 251$ does hold.

1. (a) Prove **by induction** (or by well-ordering) that $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.
   (b) Prove **by induction** (or by well-ordering) that $(5/4)^n - (3/4)^n \ge n/2$ for all integers $n \ge 1$. [*Note*: you might have to consider the cases $n = 1, 2, 3$ and $n \ge 4$ separately.]
   (c) Prove that, for all real numbers $x \ge 2$, if $(5/4)^x - (3/4)^x \ge x/2$ then $3^x + 4^x \le 5^x$. Use this and part (b) to give another proof that $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.

(a) *Basis step.* When $n = 2$ the statement to be proved is $3^2 + 4^2 \le 5^2$, which is true since $9 + 16 = 25$.

Inductive step. Assume that $3^k + 4^k \le 5^k$ holds for some integer $k \ge 2$. We want to prove that $3^{k+1} + 4^{k+1} \le 5^{k+1}$. Well, we get

$$\begin{aligned} 5^{k+1} = 5 \cdot 5^k &\ge 5(3^k + 4^k) \quad \text{from the assumption} \\ &= 5 \cdot 3^k + 5 \cdot 4^k \\ &> 3 \cdot 3^k + 4 \cdot 4^k = 3^{k+1} + 4^{k+1} \end{aligned}$$

so the inductive step is proved.

Therefore $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.

(b) *Basis step.* When $n = 1$ the statement to be proved is $\frac{5}{4} - \frac{3}{4} \ge \frac{1}{2}$ or $\frac{1}{2} \ge \frac{1}{2}$, which is true.

Inductive step. Assume that $(5/4)^k - (3/4)^k \ge k/2$ for some integer $k \ge 1$. We want to prove that $(5/4)^{k+1} - (3/4)^{k+1} \ge (k+1)/2$. From our assumption we get that $(5/4)^k \ge (3/4)^k + k/2$, so by multiplying both sides by $5/4$ we get

$$\left(\frac{5}{4}\right)^{k+1} = \frac{5}{4}\left(\frac{5}{4}\right)^k \ge \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5}{4} \cdot \frac{k}{2} = \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5k}{8} \, .$$

Thus

$$\left(\frac{5}{4}\right)^{k+1} - \left(\frac{3}{4}\right)^{k+1} \ge \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5k}{8} - \left(\frac{3}{4}\right)^{k+1} = \left(\frac{5}{4} - \frac{3}{4}\right)\left(\frac{3}{4}\right)^k + \frac{5k}{8} = \frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8} \, ,$$

Now, *if we knew that*

$$\frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8} \ge \frac{k+1}{2} \, , \tag{1}$$

then we would know that $\left(\frac{5}{4}\right)^{k+1} - \left(\frac{3}{4}\right)^{k+1} \ge \frac{k+1}{2}$, which is what we want to prove. So we need only prove (1) for all integers $k \ge 1$. Note that $\frac{1}{2}\left(\frac{3}{4}\right)^k > 0$, so $\frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8} > \frac{5k}{8}$, which means that to prove (1) we need only prove that $\frac{5k}{8} \ge \frac{k+1}{2}$. This is equivalent to $10k \ge 8k + 8$, or $2k \ge 8$, or $k \ge 4$. So we have proved (1) for all integers $k \ge 4$, which means we still have to prove (1) when $k = 1, 2$ and 3. We do this individually:

- When $k = 1$, (1) says $\frac{1}{2}\left(\frac{3}{4}\right) + \frac{5}{8} \ge 1$, which is true since $\frac{3}{8} + \frac{5}{8} = 1$.

- When $k = 2$, (1) says $\frac{1}{2}\left(\frac{9}{16}\right) + \frac{5}{4} \geq \frac{3}{2}$, which is true since $\frac{9}{32} + \frac{5}{4} = \frac{49}{32} > \frac{3}{2}$.
- When $k = 3$, (1) says $\frac{1}{2}\left(\frac{27}{64}\right) + \frac{15}{8} \geq 2$, which is true since $\frac{27}{128} + \frac{15}{8} = \frac{267}{128} > 2$.

This finishes the inductive step.

Since both the basis step and inductive step are now proved, we have proved that $(5/4)^n - (3/4)^n \geq n/2$ for all integers $n \geq 1$.

Note. Alternatively, we could have put $n = 1, 2, 3$ and 4 all into the basis step, then in the inductive step we would only need to consider the case $k \geq 4$. Or we could have handled the cases $n = 1, 2$ and 3 separately at the beginning, then use induction to prove the inequality for all integers $n \geq 4$ only, with only the case $n = 4$ in the basis step.

(c) Let x be an arbitrary real number with $x \geq 2$, and assume that $(5/4)^x - (3/4)^x \geq x/2$. We want to prove that $3^x + 4^x \leq 5^x$. Well, from $(5/4)^x - (3/4)^x \geq x/2$ we multiply both sides by 4^x to get $5^x - 3^x \geq (\frac{x}{2})4^x$, then rearrange to get $5^x \geq 3^x + (\frac{x}{2})4^x$. Since $x \geq 2$, $x/2 \geq 1$, so $5^x \geq 3^x + (\frac{x}{2})4^x \geq 3^x + 4^x$, so $3^x + 4^x \leq 5^x$ as required. Done.

Now if $n \geq 2$ is an integer, then $n \geq 1$, so from part (b) we know that $(5/4)^n - (3/4)^n \geq n/2$. Therefore, since $n \geq 2$, we know from the first part of (c) that $3^n + 4^n \leq 5^n$.

2. The sequence $b_1, b_2, \ldots$ is defined by: $b_1 = 1$, and $b_n = \left\lceil \frac{n}{b_{n-1}} \right\rceil$ for all integers $n \geq 2$.

(a) Find b_2, b_3, b_4, b_5 and b_6.

(b) Use part (a) (and more data if you need it) to guess a simple formula for b_n in terms of n. [*Hint*: do the cases of odd n and even n separately.]

(c) Use **induction** (or well-ordering) to prove your guess.

(d) Suppose the sequence $c_1, c_2, \ldots$ is defined by: $c_1 = 1, c_2 = 1$, and $c_n = \left\lceil \frac{n}{c_{n-2}} \right\rceil$ for all integers $n \geq 3$. Calculate enough terms of the sequence to enable you to see a pattern. Use that pattern to guess what c_{271} and c_{281} are. (No proof needed — yet.)

(a) We get

- $b_1 = 1$, $\quad b_2 = \lceil 2/b_1 \rceil = \lceil 2/1 \rceil = 2$,
- $b_3 = \lceil 3/b_2 \rceil = \lceil 3/2 \rceil = 2$, $\quad b_4 = \lceil 4/b_3 \rceil = \lceil 4/2 \rceil = 2$,
- $b_5 = \lceil 5/b_4 \rceil = \lceil 5/2 \rceil = 3$, $\quad b_6 = \lceil 6/b_5 \rceil = \lceil 6/3 \rceil = 2$.

(b) We could guess (maybe by further calculating that $b_7 = \lceil 7/b_6 \rceil = \lceil 7/2 \rceil = 4$ and $b_8 = \lceil 8/b_7 \rceil = \lceil 8/4 \rceil = 2$ for instance) that, for all integers $n \geq 1$,

$$b_n = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(c) *Basis step.* The formula for b_n is correct when $n = 1$ (which is odd), because $(1+1)/2 = 1 = b_1$. This will turn out to be all we need for the basis step.

Inductive step. Assume that the formula for b_n is true when $n = k$, where $k \geq 1$ is some integer. There are two cases:

- If k is even, then we are assuming that $b_k = 2$, so

$$b_{k+1} = \left\lceil \frac{k+1}{b_k} \right\rceil = \left\lceil \frac{k+1}{2} \right\rceil = \frac{k+2}{2}$$

since $k+1$ is odd, which agrees with the formula for b_{k+1} when $n = k+1$.

- If k is odd, then we are assuming that $b_k = (k+1)/2$, so

$$b_{k+1} = \left\lceil \frac{k+1}{b_k} \right\rceil = \left\lceil \frac{k+1}{(k+1)/2} \right\rceil = \lceil 2 \rceil = 2,$$

which agrees with the formula for b_{k+1} when $n = k+1$ since $k+1$ is even.

So the formula is correct when $n = k+1$ in either case. This proves the inductive step.

Therefore by induction the formula for b_n is correct for all integers $n \geq 1$.

(d) This time we get

- $c_3 = \lceil 3/1 \rceil = 3$, $c_4 = \lceil 4/1 \rceil = 4$, $c_5 = \lceil 5/3 \rceil = 2$, $c_6 = \lceil 6/4 \rceil = 2$,
- $c_7 = \lceil 7/2 \rceil = 4$, $c_8 = \lceil 8/2 \rceil = 4$, $c_9 = \lceil 9/4 \rceil = 3$, $c_{10} = \lceil 10/4 \rceil = 3$,
- $c_{11} = \lceil 11/3 \rceil = 4$, $c_{12} = \lceil 12/3 \rceil = 4$, $c_{13} = \lceil 13/4 \rceil = 4$, $c_{14} = \lceil 14/4 \rceil = 4$,
- $c_{15} = \lceil 15/4 \rceil = 4$, $c_{16} = \lceil 16/4 \rceil = 4$, $c_{17} = \lceil 17/4 \rceil = 5$, $c_{18} = \lceil 18/4 \rceil = 5$,
- $c_{19} = \lceil 19/5 \rceil = 4$, $c_{20} = \lceil 20/5 \rceil = 4$, $c_{21} = \lceil 21/4 \rceil = 6$, $c_{22} = \lceil 22/4 \rceil = 6$.

From this we guess that $c_n = 4$ whenever $n > 3$ is of the form $4k$ or $4k+3$ for some integer k, and $c_n = k+1$ whenever n is of the form $4k+1$ or $4k+2$ for some integer k. For example, $21 = 4 \cdot 5 + 1$, so $c_{21} = 5 + 1 = 6$. From this pattern we would guess that since $271 = 4 \cdot 67 + 3$, c_{271} should be **4**, while since $281 = 4 \cdot 70 + 1$, c_{281} should be $70 + 1 =$ **71**.

3. You are given the following "while" loop:

[*Pre-condition*: m is a nonnegative integer, $a = 0$, $b = 0$, $i = 0$.]

while $(i \neq m)$

1. $b := a + b + 1$
2. $a := a - 4b$
3. $i := i + 1$

end while

[*Post-condition*: $b = m(-1)^{m+1}$.]

Loop invariant: $I(n)$ is

$$i = n, \quad a = \left\{ \begin{array}{ll} -2(n+1) & \text{if } n \text{ is odd} \\ 2n & \text{if } n \text{ is even} \end{array} \right\}, \quad b = n(-1)^{n+1}.$$

(a) Prove the correctness of this loop with respect to the pre- and post-conditions.

(b) Suppose the "while" loop is as above, except that statement 2 is replaced by: $a := a - b$. Run through the loop often enough, recording the various values of a and b that result, until you can predict what the post-condition value of b will be when $m = 271$. What is your prediction? Explain.

(a) We first need to check that the loop invariant holds when $n = 0$. Since 0 is even, $I(0)$ says $i = 0$, $a = 2 \cdot 0 = 0$, and $b = 0(-1)^1 = 0$, and these are all true by the pre-conditions.

So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$ where $k < m$. We want to prove that $I(k+1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that

$$\left\{ \begin{array}{llll} i = k, & a = -2(k+1), & b = k(-1)^{k+1} = k & \text{if } k \text{ is odd}, \\ i = k, & a = 2k, & b = k(-1)^{k+1} = -k & \text{if } k \text{ is even}, \end{array} \right\}$$

and we now go through the loop.

- Step 1: $b := a + b + 1 = \left\{ \begin{array}{ll} -2(k+1) + k + 1 & \text{if } k \text{ is odd} \\ 2k - k + 1 & \text{if } k \text{ is even} \end{array} \right\}$

$$= \left\{ \begin{array}{ll} (k+1)(-1) & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{array} \right\} = (k+1)(-1)^{k+2},$$

  which agrees with the formula for b in $I(k+1)$.
- Step 2: $a := a - 4b = \left\{ \begin{array}{ll} -2(k+1) - 4(-k-1) & \text{if } k \text{ is odd} \\ 2k - 4(k+1) & \text{if } k \text{ is even} \end{array} \right\}$

$$= \left\{ \begin{array}{ll} 2(k+1) & \text{if } k+1 \text{ is even} \\ -2(k+2) & \text{if } k+1 \text{ is odd} \end{array} \right\},$$

  which agrees with the formula for a in $I(k+1)$.
- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k+1)$.

Thus $I(k+1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $b = m(-1)^{m+1}$ as required.

(b) If we set the variables to their pre-condition values of $a = 0$, $b = 0$ and $i = 0$, and run through the loop, the new values we get are $b = 0 + 0 + 1 = 1$, $a = 0 - 1 = -1$, $i = 1$. If we continue to run through the loop, and keep track of the variables in a table, here is what we get:

n	0	1	2	3	4	5	6
b	0	1	1	0	-1	-1	0
a	0	-1	-2	-2	-1	0	0
i	0	1	2	3	4	5	6

At this point (when $n = 6$) our values of b and a are back to what they were at the beginning (when $n = 0$), namely $b = a = 0$. Since the loop calculates the new values of a and b only in terms of their old values, and not in terms of n for example, the values of a and b should continue to cycle through the same values in the above table. This means that $a = b = 0$ whenever n is a multiple of 6, $a = -1$ and $b = 1$ whenever n is 1 plus a multiple of 6, and so on. Since $271 = 6 \cdot 45 + 1$, when the loop ends (at $m = i = n = 271$), we should have $b = \mathbf{1}$.

MATH 271 ASSIGNMENT 3 SOLUTIONS

1. For each integer $n \geq 1$, let $\mathcal{S}_n$ be the statement: for all sets $A, B_1, B_2, \ldots, B_n$,

$$(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_n) = A - (B_1 \cap B_2 \cap \cdots \cap B_n).$$

   (a) Prove that for all sets A, B and C, $(A - B) \cup (A - C) = A - (B \cap C)$. You may use the properties on page 272.

   (b) Prove **by induction** on n (or well ordering) that $\mathcal{S}_n$ is true for all integers $n \geq 1$.

   (a) Using the identities on page 272, we get

$$\begin{aligned}(A - B) \cup (A - C) &= (A \cap B^c) \cup (A \cap C^c) \quad \text{by \#12} \\ &= A \cap (B^c \cup C^c) \quad \text{by \#3(b) (distributive law)} \\ &= A \cap (B \cap C)^c \quad \text{by \#9(b) (De Morgan's Law)} \\ &= A - (B \cap C). \quad \text{by \#12}\end{aligned}$$

   (b) *Basis step.* When $n = 1$ the statement $\mathcal{S}_1$ says $A - B_1 = A - B_1$, which is obviously true.

   Inductive step. Assume that the statement $\mathcal{S}_k$ is true for all sets $A, B_1, B_2, \ldots, B_k$, where $k \geq 1$ is an integer. We want to prove that the statement $\mathcal{S}_{k+1}$ is true for all sets $A, B_1, B_2, \ldots, B_{k+1}$. So let $A, B_1, B_2, \ldots, B_{k+1}$ be sets. Then we want to prove that

$$(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_{k+1}) = A - (B_1 \cap B_2 \cap \cdots \cap B_{k+1}). \qquad (2)$$

   Well,

$$\begin{aligned}&(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_{k+1}) \\ &\quad= \big((A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_k)\big) \cup (A - B_{k+1}) \\ &\quad= \big(A - (B_1 \cap B_2 \cap \cdots \cap B_k)\big) \cup (A - B_{k+1}) \quad \text{by assumption} \\ &\quad= A - (B_1 \cap B_2 \cap \cdots \cap B_k \cap B_{k+1}) \quad \text{by part (a)} \\ &\quad= A - (B_1 \cap B_2 \cap \cdots \cap B_{k+1})\end{aligned}$$

   which proves (1). This finishes the inductive step. Therefore by induction, $\mathcal{S}_n$ is true for all integers $n \geq 1$.

2. A sequence is called an *A-sequence* if every term is equal to 1, 2 or 3, and no two consecutive terms in the sequence are equal. Also, an A-sequence is called a *B-sequence* if the first and last terms are equal. So for example, 23121 is an A-sequence which is not a B-sequence, 23212 is both an A-sequence and a B-sequence, and 23122 is neither. For each integer $n \geq 2$, let a_n be the number of A-sequences of length n and let b_n be the number of B-sequences of length n.

   (a) Show that $a_n = 3 \cdot 2^{n-1}$ for all integers $n \geq 2$.

(b) Prove combinatorially that $b_n = a_{n-1} - b_{n-1}$ for all integers $n \geq 3$. [*Hint*: how can you make a B-sequence of length n from an A-sequence of length $n-1$?]

(c) Show $b_2 = 0$, and then use parts (a) and (b) to find b_3, b_4 and b_5.

(d) Use your answers to part (c) (and more if you need them) to guess a simple formula for b_n. [*Hint*: how far away is b_n from a nearby power of 2?]

(e) Use parts (a) and (b) to prove your formula in (d) **by induction** (or well ordering) for all integers $n \geq 2$.

(a) We count how many ways there are to make an A-sequence of length n. There are 3 choices for the first term (1, 2 or 3). After that each term can be any of 1, 2 or 3 except for what the previous term is, so there are 2 choices for each of the $n-1$ terms after the first. By the multiplication rule, the number of A-sequences of length n must be $a_n = 3 \cdot 2 \cdot 2 \cdots 2 = 3 \cdot 2^{n-1}$.

(b) To make a B-sequence of length n, we could take an A-sequence of length $n-1$ and add an nth term equal to the first term of the A-sequence. But for this to be allowed, the last two terms of the resulting sequence could not be equal, which means that the first and last terms of the original A-sequence cannot be equal. Thus we must start with an A-sequence which is **not** a B-sequence. There are a_{n-1} A-sequences of length $n-1$, and b_{n-1} of these are B-sequences, so there are $a_{n-1} - b_{n-1}$ A-sequences of length $n-1$ which are not B-sequences. Each of them can be converted to a B-sequence of length n as described above, and every B-sequence of length n will arise this way. Therefore the number b_n of B-sequences of length n must equal $a_{n-1} - b_{n-1}$.

(c) A B-sequence of length 2 must start and end with the same symbol, so both of its terms must be equal, but this is not allowed. So there are no B-sequences of length 2, that is, $b_2 = 0$. Now from parts (a) and (b) we get

- $b_3 = a_2 - b_2 = 3 \cdot 2^1 - 0 = 6$,
- $b_4 = a_3 - b_3 = 3 \cdot 2^2 - 6 = 12 - 6 = 6$,
- $b_5 = a_4 - b_4 = 3 \cdot 2^3 - 6 = 24 - 6 = 18$.

(d) Since

- $b_2 = 0 = 2 - 2 = 2^1 - 2$,
- $b_3 = 6 = 4 + 2 = 2^2 + 2$,
- $b_4 = 6 = 8 - 2 = 2^3 - 2$, and
- $b_5 = 18 = 16 + 2 = 2^4 + 2$,

we can guess that $b_n = 2^{n-1} - 2$ for n even and $b_n = 2^{n-1} + 2$ for n odd, which could be written as:

$$b_n = 2^{n-1} - 2(-1)^n \quad \text{for all integers } n \geq 2.$$

(e) *Basis step.* We already know that the formula $b_n = 2^{n-1} - 2(-1)^n$ is correct for $n = 2$, since $b_2 = 0$.

Inductive step. Assume that $b_k = 2^{k-1} - 2(-1)^k$ is true for some integer $k \geq 2$. Then

$$\begin{aligned} b_{k+1} &= a_k - b_k \quad \text{by part (b)} \\ &= 3 \cdot 2^{k-1} - (2^{k-1} - 2(-1)^k) \quad \text{by part (a) and by assumption} \\ &= 2 \cdot 2^{k-1} + 2(-1)^k = 2^k - 2(-1)^{k+1}, \end{aligned}$$

which proves that the formula for b_n is true for $n = k + 1$.

Therefore by induction, the formula for b_n is true for all integers $n \geq 2$.

Bonus question. Such a nice answer cries out for a nicer proof! Can you find a combinatorial proof that $b_n = 2^{n-1} - 2(-1)^n$? Maybe something like the proof in part (a), or something involving all subsets of an $(n-1)$-element set? But it would have to be rather clever, to account for the ± 2 in the formula. If you think you can get somewhere with this problem or have some ideas, tell your professor or TA. [*Warning*: neither of the professors in the course knows how to do this problem!]

3. A *balanced* subset of integers is a subset that has the same number of even integers as odd integers. For example, the subset $\{1, 2, 4, 9\}$ is balanced, but $\{1, 2, 3\}$ is not. For each positive integer n, let b_n be the number of balanced subsets of $\{1, 2, \ldots, 2n\}$.

(a) Prove combinatorially that for all positive integers n,

$$b_n = \sum_{i=0}^{n} \binom{n}{i}^2 = 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + 1.$$

(b) Use part (a) to calculate b_1, b_2 and b_3.

(c) Use your answers to part (b) to guess a formula for b_n which is a single binomial coefficient involving n. [*Hint*: write out Pascal's Triangle (page 359) up to $n = 6$.]

(d) Prove your formula in part (c) combinatorially for all integers $n \geq 1$. [*Hint*: what if you construct a balanced subset of $\{1, 2, \ldots, 2n\}$ by choosing which even integers to put in your subset and which odd integers to *leave out* of your subset? How many integers would you choose altogether?]

(a) We count the balanced subsets of $\{1, 2, \ldots, 2n\}$ by counting how many balanced subsets there are of each size separately and adding these numbers together. There is only one way to choose a balanced subset of no elements, namely the empty set. There are $\binom{n}{1}$ ways to choose one even integer from $\{1, 2, \ldots, 2n\}$ and $\binom{n}{1}$ ways to choose one odd integer from $\{1, 2, \ldots, 2n\}$, so by the multiplication rule there are $\binom{n}{1}\binom{n}{1} = \binom{n}{1}^2$ ways to choose a balanced subset of size 2. In general, for each $k \in \{0, 1, \ldots, n\}$, there are $\binom{n}{k}$ ways to choose k even integers from $\{1, 2, \ldots, 2n\}$ and $\binom{n}{k}$ ways to choose k odd integers from $\{1, 2, \ldots, 2n\}$, so by the multiplication rule there are $\binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$ ways to choose a balanced subset of size $2k$. Therefore by the addition rule, there are

$$\sum_{k=0}^{n} \binom{n}{k}^2 = 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + 1$$

balanced subsets of $\{1, 2, \ldots, 2n\}$ altogether, so this must equal b_n.

(b) From part (a),

- $b_1 = \binom{1}{0}^2 + \binom{1}{1}^2 = 1 + 1 = 2;$
- $b_2 = \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 1 + 2^2 + 1 = 6;$
- $b_3 = \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 1 + 3^2 + 3^2 + 1 = 20.$

(c) We know that $b_1 = 2 = \binom{2}{1}$. Looking at the fourth row $(1, 4, 6, 4, 1)$ of Pascal's Triangle we see that $\binom{4}{2} = 6 = b_2$. Looking at the sixth row $(1, 6, 15, 20, 15, 6, 1)$ of Pascal's Triangle we see that $\binom{6}{3} = 20 = b_3$. So we guess that $b_n = \binom{2n}{n}$ for every positive integer n.

(d) We can make a balanced subset of $\{1, 2, \ldots, 2n\}$ by choosing k even numbers from this set, and then choosing $n - k$ odd numbers to leave out (so that the remaining k odd numbers from $\{1, 2, \ldots, 2n\}$ are added in with the k even numbers to make a balanced set). Thus we are choosing $k + (n - k) = n$ numbers from $\{1, 2, \ldots, 2n\}$ to create the balanced set this way. Moreover, if we choose any set S of n numbers from $\{1, 2, \ldots, 2n\}$, we can make a balanced subset of $\{1, 2, \ldots, 2n\}$ out of S in the above way: if S includes (say) k even integers, then it must have $n - k$ odd integers, so there must be exactly k odd integers from $\{1, 2, \ldots, 2n\}$ which are *not* included in S; so just keep all the even integers in S plus all the odd integers in $\{1, 2, \ldots, 2n\}$ which are not in S, and we get a balanced subset of $\{1, 2, \ldots, 2n\}$. Thus the number of balanced subsets of $\{1, 2, \ldots, 2n\}$ is the same as the number of n-element subsets of $\{1, 2, \ldots, 2n\}$, which is $\binom{2n}{n}$. Thus $b_n = \binom{2n}{n}$.

Note. This fact is in Exercise 19, page 362.

1. If $f : X \to X$ is a function, define $f^2(x)$ to be $(f \circ f)(x)$, and inductively define $f^k(x) = (f \circ f^{k-1})(x)$ for each integer $k \geq 3$. (So $f^3(x) = (f \circ f^2)(x) = f(f(f(x)))$ for instance.) We also define $f^1(x)$ to be $f(x)$.

   Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by: for all $n \in \mathbb{Z}$, $f(n) = n + 1 + 2(-1)^n = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$

   (a) Find $f^2(n)$, $f^3(n)$, and $f^4(n)$.

   (b) Use part (a) (and more data if you need it) to guess a fairly simple formula for $f^k(n)$ for any positive integer k. (You may need to consider k odd and k even separately.)

   (c) Use induction on k (or well ordering) to prove your guess.

   (d) Use your formula for $f^k(n)$ to find $f^{2008}(271)$.

   (e) Define $g : \mathbb{Z} \to \mathbb{Z}$ by: for all $n \in \mathbb{Z}$, $g(n) = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ 1-n & \text{if } n \text{ is odd.} \end{cases}$

   Calculate $g^2(n)$, $g^3(n)$, and $g^4(n)$, and use them (and more data if you need it) to predict what $g^{2008}(271)$ is.

   (a) We get

$$\begin{aligned} f^2(n) &= f(f(n)) = f(n+1+2(-1)^n) \\ &= (n+1+2(-1)^n) + 1 + 2(-1)^{n+1+2(-1)^n} \\ &= n+2+2(-1)^n + 2(-1)^{n+1} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n+2 \qquad \text{since } n \text{ and } n+1 \text{ are of opposite parity,} \end{aligned}$$

$$f^3(n) = f(f^2(n)) = f(n+2) = (n+2) + 1 + 2(-1)^{n+2} = n+3+2(-1)^n,$$

$$\begin{aligned} f^4(n) &= f(f^3(n)) = f(n+3+2(-1)^n) \\ &= (n+3+2(-1)^n) + 1 + 2(-1)^{n+3+2(-1)^n} \\ &= n+4+2(-1)^n + 2(-1)^{n+3} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n+4 \qquad \text{since } n \text{ and } n+3 \text{ are of opposite parity.} \end{aligned}$$

   (b) From part (a) we would guess that

$$f^k(n) = \begin{cases} n+k+2(-1)^n & \text{if } k \text{ is odd,} \\ n+k & \text{if } k \text{ is even.} \end{cases}$$

   (c) *Basis step.* Our guessed formulas for $f^k(n)$ are true for $k = 1, 2, 3$ and 4, by part (a).

   Inductive step. Assume that our guessed formula is true for some integer $k = \ell \geq 1$. We want to prove that our formula is true when $k = \ell + 1$. We do this in two cases:

   Case (i): ℓ is even. So we assume that $f^\ell(n) = n + \ell$, and we want to prove that $f^{\ell+1}(n) = n + \ell + 1 + 2(-1)^n$.

Well, we get

$$f^{\ell+1}(n) = f(f^{\ell}(n)) = f(n+\ell) = n+\ell+1+2(-1)^{n+\ell} = n+\ell+1+2(-1)^n,$$

since ℓ is even, so the inductive step works in this case.

Case (ii): ℓ is odd. This time we assume that $f^{\ell}(n) = n+\ell+2(-1)^n$, and we want to prove that $f^{\ell+1}(n) = n+\ell+1$. We get

$$\begin{aligned} f^{\ell+1}(n) &= f(f^{\ell}(n)) = f(n+\ell+2(-1)^n) \\ &= (n+\ell+2(-1)^n)+1+2(-1)^{n+\ell+2(-1)^n} \\ &= n+\ell+1+2(-1)^n+2(-1)^{n+\ell} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n+\ell+1 \qquad \text{since } n \text{ and } n+\ell \text{ are of opposite parity,} \end{aligned}$$

so the inductive step works in this case too. Therefore the guessed formula is true for all integers $k \geq 1$.

(d) By the formula, since 2008 is even, $f^{2008}(271) = 271+2008 = 2279$.

(e) We get

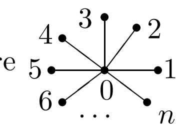
$$\begin{aligned} g^2(n) &= g(g(n)) = \begin{cases} g(n+3) & \text{if } n \text{ is even,} \\ g(1-n) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} 1-(n+3) & \text{if } n \text{ is even (since } n+3 \text{ is odd),} \\ (1-n)+3 & \text{if } n \text{ is odd (since } 1-n \text{ is even).} \end{cases} \\ &= \begin{cases} -n-2 & \text{if } n \text{ is even,} \\ -n+4 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$\begin{aligned} g^3(n) &= g(g^2(n)) = \begin{cases} g(-n-2) & \text{if } n \text{ is even,} \\ g(-n+4) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} (-n-2)+3 & \text{if } n \text{ is even (since } -n-2 \text{ is even),} \\ 1-(-n+4) & \text{if } n \text{ is odd (since } -n+4 \text{ is odd).} \end{cases} \\ &= \begin{cases} -n+1 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$\begin{aligned} g^4(n) &= g(g^3(n)) = \begin{cases} g(-n+1) & \text{if } n \text{ is even,} \\ g(n-3) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} 1-(-n+1) & \text{if } n \text{ is even (since } -n+1 \text{ is odd),} \\ (n-3)+3 & \text{if } n \text{ is odd (since } n-3 \text{ is even).} \end{cases} \\ &= n \qquad \text{for all } n \in \mathbb{Z}. \end{aligned}$$

Now $g^5(n) = g(g^4(n)) = g(n)$, and so $g^6(n) = g(g^5(n)) = g(g(n)) = g^2(n)$, and so on; the formulas for $g^k(n)$ will cycle through the above four functions forever. In particular, since 2008 is a multiple of 4, $g^{2008}(n)$ will equal n for all n, so $g^{2008}(271) = 271$.

2. For each integer $n \geq 2$, let S_n be the "star-like" graph shown at the right, where there are $n+1$ vertices altogether, including the one in the middle.



(a) Find a formula (in terms of n) for the number of paths of length 2 in S_n. [*Hint*: how many such paths start at each vertex?]

(b) Find a formula (in terms of n) for the number of walks of length 2 in S_n.

(c) Write out the $(n+1) \times (n+1)$ adjacency matrix M_n of S_n in general. Then find M_n^2, and explain what it has to do with your answer to part (b).

(d) Prove that for any simple graph G, the number of walks in G of length 2 is always even. [*Hint*: how can you pair up the walks?]

(a) Since paths cannot repeat vertices, there are no paths of length 2 starting at vertex 0, because once we go from 0 to one of the other vertices we are stuck. If we start at one of the other vertices, say at vertex 1, then we must go to vertex 0, and from there we have $n-1$ choices for the third vertex, namely any vertex except vertex 0 or 1. So there are $n-1$ paths of length 2 starting from vertex 1. By symmetry there are $n-1$ paths of length 2 starting from any of the vertices 1 to n, so there are $n(n-1)$ paths of length 2 in S_n altogether.

(b) For walks we are allowed to repeat vertices, so if we start at vertex 0 we can go to any of the vertices 1 to n, and then we must go back to 0. So there are n walks of length 2 starting at vertex 0. If we start at vertex 1 instead, then we must go to vertex 0, and then we can go to any of the vertices 1 to n, so there are n walks of length 2 starting at vertex 1. Again by symmetry there are n walks of length 2 starting from any of the vertices 1 to n, so there are $n + n(n) = n + n^2$ walks of length 2 in S_n.

(c) If we order the rows and columns in the natural way (with the vertices in the order $0, 1, 2, \ldots, n$), we will get

$$M_n = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \text{and so} \qquad M_n^2 = \begin{bmatrix} n & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

The matrix M_n^2 tells us how many walks of length 2 the graph S_n has between all possible pairs of vertices. Since the sum of all the entries in M_n^2 is $n + n^2$, this must be the number of walks of length 2 in S_n, which agrees with our answer in part (b).

(d) In a simple graph G, every walk of length 2 is either of the form abc, where a, b, c are three different vertices, or of the form aba where a and b are different vertices. Walks of the first kind can be paired up by pairing each walk abc with the walk cba. Walks of the second kind can be paired up by pairing each walk aba with the walk bab. Thus all walks of length 2 are paired up, so there must be an even number of them altogether.

3. For each integer $n \geq 3$, let G_n be the graph whose vertices are all two-element subsets of $\{1, 2, \ldots, n\}$, and with edges defined as follows: for any vertices A and B of G_n (so A and B are two-element subsets of $\{1, 2, \ldots, n\}$), A and B are adjacent if and only if $N(A \cap B) = 1$ (where $N(X)$ is the number of elements in the set X).

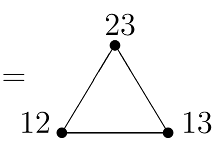
   (a) Draw the graphs G_3 and G_4. [You can label the vertex $\{i, j\}$ as just ij if you like.]

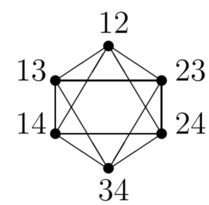
   (b) For each integer $n \geq 3$, find and prove formulas (in terms of n) for the number of vertices in G_n, the degree of each vertex, and the number of edges in G_n.

   (c) For which n does G_n have an Euler circuit? Explain.

   (d) For which n does G_n have a Hamiltonian circuit? Explain. [*Hint*: induction on n.]

(a) For G_3 the vertices are all two-element subsets of $\{1, 2, 3\}$, so they are 12, 13 and 23 (writing the subset $\{1, 2\}$ as just 12 for instance). For G_4 the vertices are similarly 12, 13, 14, 23, 24 and 34. We get

$G_3 =$ 

$G_4 =$ 

(b) The number of vertices in G_n is $\binom{n}{2}$, the number of 2-element subsets of $\{1, 2, \ldots, n\}$.

The degree of the vertex 12 of G_n is the number of 2-element subsets of $\{1, 2, \ldots, n\}$ which contain either 1 or 2 (but not both). There are $n - 2$ 2-element subsets that contain 1 but not 2, and $n-2$ 2-element subsets that contain 2 but not 1, so the degree of 12 must be $2(n - 2) = 2n - 4$. By symmetry the degree of every vertex of G_n is $2n - 4$.

From above, the sum of the degrees of the vertices of G_n must be the number of vertices times the degree of each vertex, which is $\binom{n}{2}(2n-4)$. This is twice the number of edges, so the number of edges in G_n must be

$$\frac{1}{2}\binom{n}{2}(2n - 4) = \binom{n}{2}(n - 2) = \frac{n(n - 1)(n - 2)}{2} .$$

(c) It is clear that G_n is connected, because if ab and cd are arbitrary nonadjacent vertices of G_n, then ab, ad, cd is a walk in G_n from ab to cd.

Now, since the degree of each vertex of G_n is $2n - 4$ which is even, G_n will have an Euler circuit for all integers $n \geq 3$.

(d) We prove by induction on n that G_n has a Hamiltonian circuit for each integer $n \geq 3$.

Basis step. It is easy to see from the graph that G_3 has a Hamiltonian circuit, for example $12, 13, 23, 12$ is a Hamiltonian circuit in G_3.

Inductive step. Assume that G_k has a Hamiltonian circuit for some integer $k \geq 3$. This means that there is a circuit in G_k containing each vertex exactly once (except that the first vertex equals the last vertex). We can start this circuit at any vertex we like, so let's say we start it at the vertex 12. The second vertex in the circuit must be a 2-element subset of $\{1, 2, \ldots, k\}$ which contains either 1 or 2 (but not both), so by symmetry we can assume it is $1k$. So the Hamiltonian circuit starts off $12, 1k$ and so on, eventually ending back at 12 after going through each vertex of G_k exactly once.

We want to find a Hamiltonian circuit in G_{k+1}, and we will use the Hamiltonian circuit in G_k to do this. The vertices of G_{k+1} which are not vertices of G_k are just the 2-element subsets of $\{1, 2, \ldots, k+1\}$ which contain $k+1$, so they are $1(k+1), 2(k+1), \ldots, k(k+1)$. All the edges in G_k are still in G_{k+1}, so we replace the edge $12, 1k$ of the Hamiltonian circuit in G_k by the path $12, 1(k+1), 2(k+1), \ldots, k(k+1), 1k$, and then we will get a Hamiltonian circuit in G_{k+1}. Note that this is allowed, since each two consecutive vertices of the path $12, 1(k+1), 2(k+1), \ldots, k(k+1), 1k$ have an element in common, so they are adjacent in G_{k+1}.

For example, the Hamiltonian circuit $12, 13, 23, 12$ in G_3 would be used to make the Hamiltonian circuit $12, 14, 24, 34, 13, 23, 12$ in G_4, by replacing the edge $12, 13$ by the path $12, 14, 24, 34, 13$ in G_4, which contains all the vertices in G_4 which are not in G_3.

This completes the inductive step. By induction, G_n has a Hamiltonian circuit for every integer $n \geq 3$.