

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATHEMATICS 211 — L60 Summer 2009
MIDTERM EXAM [July 30, 2009 (Thursday)]

Time: 75 minutes. PLEASE write your Name on the very last page.

NO CALCULATORS.

Total Marks = 100. Work all problems. Marks are shown in brackets.

Student ID: _____

[Marks]

1. Let
$$\begin{cases} 3x_1 - 5x_2 - 7x_3 - 3x_4 = 9 \\ 2x_1 - 5x_2 + 2x_3 - 7x_4 = 11 \\ -x_1 + 2x_2 + x_3 + 2x_4 = -4 \end{cases} .$$

[2] (a) Write down the augmented matrix $K = [A|B]$ of the system;

[14] (b) Find the reduced row-echelon form R of $K = [A|B]$ and an invertible matrix U of size 3×3 such that the product $U K = R$;

- (c) Use the reduced row-echelon form R of $K = [A|B]$ from part (b) to find the general solution to this system. Give a particular solution and the general solution to the associated homogeneous system with basic solutions. (Your answers should be in matrix form)
- [14]

[8]

2. (a) Write the following system in matrix form and use a matrix inverse to solve it:

$$\begin{cases} 7x + 5y = 3 \\ 3x + 4y = 5 \end{cases} .$$

[4]

- (b) Find A if $(A^{-1} - 2I_2)^T = \begin{bmatrix} 5 & 3 \\ 5 & 0 \end{bmatrix}$.

- [18] 3. Use row operations to find the inverse A^{-1} of $A = \begin{bmatrix} 7 & 3 & 10 \\ 5 & 2 & 7 \\ 3 & 1 & 5 \end{bmatrix}$.

- [10] 4. Express the matrix $A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$ as a product of elementary matrices.

5. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation of vectors in \mathbb{R}^2 such that $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Write your answers in the space provided.

(a) Express the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

[4] where $a = \underline{\hspace{2cm}}$ and $b = \underline{\hspace{2cm}}$. (a and b depend on x and y)

[4] (b) Find a formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$:

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= a T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + b T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) \\ &= \underline{\hspace{2cm}} \begin{bmatrix} \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \end{bmatrix} + \underline{\hspace{2cm}} \begin{bmatrix} \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \end{bmatrix} = \begin{bmatrix} \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} \end{bmatrix}; \end{aligned}$$

[2] (c) Use the formula from part (b) to find

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \end{bmatrix};$$

[2] (d) The matrix A_T of T is $A_T = \begin{bmatrix} \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \end{bmatrix}$;

[2] (e) Is T invertible? Yes or no: . If Yes, then its inverse transformation T^{-1} is induced by the matrix $A_{T^{-1}} = \begin{bmatrix} \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \end{bmatrix}$.

6. Let $P = \begin{bmatrix} 2/5 & 4/5 \\ 3/5 & 1/5 \end{bmatrix}$ be the transition matrix of a Markov chain that starts in state 1.

[6] (a) What is the probability that the chain is in state 2 after 2 transitions?

[10] (b) Explain why this chain is regular and find the steady-state vector for the chain.

Name:	Student ID:	Marks:
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