Applied Mathematics 605 Assignment #4

due date: December 11, 2009

1. Apply the method of harmonic balance to the equation

$$\ddot{x} + x - \alpha x^2 = 0, \quad \alpha > 0,$$

using the approximate form of solution $x = c + a\cos\omega t$ to show that for α small,

$$\omega^2 = 1 - 2\alpha c, \quad c = \frac{1}{2\alpha} - \frac{1}{2\alpha}\sqrt{1 - 2\alpha^2 a^2}.$$

Deduce the frequency-amplitude relation

$$\omega = \sqrt[4]{1 - 2\alpha^2 a^2}, \quad a < \frac{1}{\sqrt{2}\alpha}.$$

2. If $h(t) = (\cos t + \sin t)/(2 + \sin t - \cos t)$, consider the differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & h(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $\Phi(t)$ be a fundamental solution, and since $h(t)$ has minimal period 2π, we know $\Phi(2\pi) = \Phi(0)C$ for some matrix C. Find the eigenvalues of C.

3. A two-species population model is described by the system

$$\begin{aligned} \dot{x}_1 &= -vx_1 + a(t)x_1, \\ \dot{x}_2 &= -vx_2 + ka(t)x_1. \end{aligned}$$

Here $a(t + 2\pi) = a(t)$ for all t, and v and k are positive constants. find the characteristic exponents and show that there is a periodic solution of period 2π if and only if

$$\int_0^{2\pi} a(t)\,dt = 2v\pi.$$