

Chapter 5

Riemannian Manifolds

1 Tensors

Let M be a manifold of dimension n . $C^\infty(M)$ the ring of smooth functions on M , $= \text{Der}C^\infty(M)$ Recall that the space $\mathfrak{X}(M)$ of vector fields and the space $\Omega^1(M)$ of 1-forms on M are modules over $C^\infty(M)$. This means that vector fields and 1-forms can be multiplied by functions in $C^\infty(M)$.

Definition 1 A covariant k -tensor field on M is a k -linear map T from $\mathfrak{X}(M)$ to $C^\infty(M)$ such that, for every $X_1, \dots, X_k \in \mathfrak{X}(M)$, $i = 1, \dots, k$, and $f \in C^\infty(M)$,

$$T(X_1, \dots, fX_i, \dots, X_k) = fT(X_1, \dots, X_i, \dots, X_k).$$

Definition 2 A contravariant l -tensor field on M is a l -linear map T from $\Omega^1(M)$ to $C^\infty(M)$ such that, for every $\omega_1, \dots, \omega_l \in \Omega^1(M)$, $i = 1, \dots, l$, and $f \in C^\infty(M)$,

$$T(\omega_1, \dots, f\omega_i, \dots, \omega_l) = fT(\omega_1, \dots, \omega_i, \dots, \omega_l). \quad (1)$$

Definition 3 A mixed tensor field of type (k, l) is a map

$$T : (X^1, \dots, X^k, \omega_1, \dots, \omega_l) \mapsto T(X^1, \dots, X^k, \omega_1, \dots, \omega_l) \quad (2)$$

which is k -linear in vector fields X^1, \dots, X^k , l -linear in forms $\omega_1, \dots, \omega_l$ and such that,

$$T(X^1, \dots, f_i X_i, \dots, X^k, \omega_1, \dots, f_j \omega_j, \dots, \omega_l) = f_i f_j T(X^1, \dots, X_i, \dots, X^k, \omega_1, \dots, \omega_j, \dots, \omega_l) \quad (3)$$

Differential k -forms on M are examples of covariant k -tensor field on M . Vector fields on M are examples of contravariant 1-tensors on M . A vector field X on M associates to a 1-form ω on M a function $\omega(X)$ on M . Mixed tensor fields of type $(k, 1)$ are often referred to as covariant k -tensor fields with values in $\mathfrak{X}(M)$. Examples of mixed tensors will be given in subsequent sections.

The condition (1) implies that, for every $x \in M$, a covariant k -tensor field T on M defines a k -linear map $T(x)$ from $T_x M$ to \mathbb{R} such that

$$T(X_1, \dots, X_k)(x) = T(x)(X_1(x), \dots, X_k(x)).$$

Similarly, a contravariant l -tensor field T defines an l -linear map from T_x^*M to \mathbb{R} such that

$$T(\omega_1, \dots, \omega_l)(x) = T(x)(\omega_1(x), \dots, \omega_l(x)).$$

Moreover, a mixed tensor field T of type (k, l) gives a map $T(x)$ such that

$$T(x) : (X^1(x), \dots, X^k(x), \omega_1(x), \dots, \omega_l(x))) \mapsto T(X^1, \dots, X^k, \omega_1, \dots, \omega_l)(x).$$

In terms of local coordinates q^1, \dots, q^k on U , a tensor field of type (k, l) is given as

$$T|_U = \sum_{i_1, \dots, i_l=1}^n \sum_{j_1, \dots, j_k=1}^n T_{j_1 \dots j_k}^{i_1 \dots i_l} dq^{j_1} \dots dq^{j_k} \frac{\partial}{\partial q^{i_1}} \dots \frac{\partial}{\partial q^{i_l}},$$

where the components $T_{j_1 \dots j_k}^{i_1 \dots i_l}$ are functions on U . If $\psi : U \rightarrow \mathbb{R}^n$ is another chart with coordinates p^i , for $i = 1, \dots, n$, then

$$T|_U = \sum_{i'_1, \dots, i'_l=1}^n \sum_{j'_1, \dots, j'_k=1}^n T_{j'_1 \dots j'_k}^{i'_1 \dots i'_l} dp^{j'_1} \dots dp^{j'_k} \frac{\partial}{\partial p^{i'_1}} \dots \frac{\partial}{\partial p^{i'_l}}$$

$dq^i = \sum_{j=1}^n \frac{\partial q^i}{\partial p^j} dp^j$ and

$$T_{j'_1 \dots j'_k}^{i'_1 \dots i'_l} = \sum_{j_1, \dots, j_k=1}^n T_{j_1 \dots j_k}^{i_1 \dots i_l} \frac{\partial q^{j_1}}{\partial p^{j'_1}} \dots \frac{\partial q^{j_k}}{\partial p^{j'_k}} \frac{\partial p^{i'_1}}{\partial q^{i_1}} \dots \frac{\partial p^{i'_l}}{\partial q^{i_l}}.$$

2 Riemannian metric

The Euclidean space \mathbb{R}^n is endowed with the Euclidean scalar product, which we will denote by $g_{\mathbb{R}^n}$, such that, for $\vec{u} = (u^1, \dots, u^n)$ and $\vec{v} = (v^1, \dots, v^n)$

$$g_{\mathbb{R}^n}(\vec{u}, \vec{v}) = \sum_{i=1}^n u^i v^i.$$

Generalization of the notion of the scalar product on \mathbb{R}^n leads to the notion of a Riemannian metric on a manifold.

Definition 4 A Riemannian metric on M is a covariant 2-tensor g on M such that, for every $X \in \mathfrak{X}(M)$ and $x \in M$,

$$g(X, X) \geq 0 \text{ and } g(X, X)(x) = 0 \text{ only if } X(x) = 0. \quad (4)$$

The property (4) is usually expressed by saying that g is positive definite. The number

$$\|X(x)\| = \sqrt{g(X, X)(x)} = \sqrt{g(x)(X(x), X(x))}$$

is interpreted as the length of the vector $X(x)$. In terms of coordinates q^1, \dots, q^n on U ,

$$g = \sum_{i,j=1}^n g_{ij} dq^i dq^j,$$

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial q^i},$$

and

$$\|X\| = \sqrt{g(X, X)} = \sqrt{\sum_{i,j=1}^n g_{ij} X^i X^j}.$$

A manifold M endowed with a Riemannian metric is called a *Riemannian manifold*.

Let $\gamma : [a, b] \rightarrow M$ be a curve in M . The derived map of γ is $T\gamma : T[a, b] \rightarrow TM$. Since $[a, b]$ is an interval in \mathbb{R} , $T[a, b] = [a, b] \times \mathbb{R}$. The map $\dot{\gamma} : [a, b] \rightarrow TM$, such that

$$\dot{\gamma}(t) = T\gamma(t, 1)$$

for every $t \in [a, b]$, is called the tangent vector of the curve γ .

Definition 5 *The length of a curve $\gamma : [a, b] \rightarrow M$ is given by*

$$L[\gamma] = \int_a^b \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

If $\gamma[a, b]$ is contained in the domain U of a coordinate chart with coordinates q^1, \dots, q^n , then we write $q^i(t) = q^i(\gamma(t))$ and $\dot{q}^i(t) = \frac{d}{dt} q^i(t)$. With this notation, $\dot{\gamma}(t) = (\dot{q}^1(t), \dots, \dot{q}^n(t))$ and

$$L[\gamma] = \int_a^b \sqrt{\sum_{i,j=1}^n g_{ij}(q^1(t), \dots, q^n(t)) \dot{q}^i(t) \dot{q}^j(t)} dt.$$

Definition 6 *A curve $\gamma : [a, b] \rightarrow M$ is a geodesic of the Riemannian metric g on M if it is a critical point of the integral $L[\gamma]$ under variations of γ which preserve the endpoints.*

We are going to derive differential equations for a geodesics contained in the domain U of a coordinate chart on M . As we have said $\gamma(t) = (q^1(t), \dots, q^n(t))$. We denote by

$$\gamma(t) + \delta\gamma(t) = (q^1(t) + \delta q^1(t), \dots, q^n(t) + \delta q^n(t))$$

a variation of γ . The assumption that the endpoints are fixed implies that

$$\delta q^1(a) = \dots = \delta q^n(a) = 0 \text{ and } \delta q^1(b) = \dots = \delta q^n(b) = 0.$$

Moreover, we have

$$\begin{aligned}\delta(\dot{q}^1(t), \dots, \dot{q}^n(t)) &= \delta\dot{\gamma}(t) = \frac{d}{dt}(\gamma(t) + \delta\gamma(t)) - \dot{\gamma}(t) = \frac{d}{dt}(\delta q^1(t), \dots, \delta q^n(t)) \\ &= \left(\frac{d}{dt}\delta q^1(t), \dots, \frac{d}{dt}\delta q^n(t)\right).\end{aligned}$$

In other words,

$$\delta\dot{q}^i(t) = \frac{d}{dt}\delta q^i(t), \text{ for } i = 1, \dots, n.$$

Hence,

$$\begin{aligned}\delta L[\gamma] &= \int_a^b \delta \left(\sqrt{\sum_{i,j=1}^n g_{ij}(q^1(t), \dots, q^n(t)) \dot{q}^i(t) \dot{q}^j(t)} \right) dt \\ &\quad \text{The} \\ &= \int_a^b \left(2 \sqrt{\sum_{i,j=1}^n g_{pq}(q^1(t), \dots, q^n(t)) \dot{q}^p(t) \dot{q}^q(t)} \right)^{-1} \left(\delta \sum_{i,j=1}^n g_{ij}(q^1(t), \dots, q^n(t)) \dot{q}^i(t) \dot{q}^j(t) \right) dt \\ &= \int_a^b \left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n \left(2g_{ij} \dot{q}^i \delta \dot{q}^j + \sum_{k=1}^n \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right) dt \\ &= \int_a^b \left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n \left(2g_{ij} \dot{q}^i \frac{d}{dt} \delta q^j + \sum_{k=1}^n \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right) dt \\ &= \int_a^b \frac{d}{dt} \left[\left(2 \sqrt{\sum_{i,j=1}^n g_{ij} \dot{q}^i \dot{q}^j} \right)^{-1} \sum_{i,j=1}^n 2g_{ij} \dot{q}^i \delta q^j \right] dt + \\ &\quad - \int_a^b \left\{ \frac{d}{dt} \left[\left(2 \sqrt{\sum_{i,j=1}^n g_{ij} \dot{q}^i \dot{q}^j} \right)^{-1} \sum_{i,j=1}^n 2g_{ij} \dot{q}^i \right] \delta q^j - \left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j,k=1}^n \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right\} \\ &= \left[\left(2 \sqrt{\sum_{i,j=1}^n g_{ij} \dot{q}^i \dot{q}^j} \right)^{-1} \sum_{i,j=1}^n 2g_{ij} \dot{q}^i \delta q^j \right]_{t=a}^{t=b} \\ &\quad - \int_a^b \left\{ \frac{d}{dt} \left[\left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n 2g_{ij} \dot{q}^i \right] \delta q^j - \left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j,k=1}^n \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right\}\end{aligned}$$

The boundary term vanishes because $\delta q^i(a) = \delta q^i(b)$ for $i = 1, \dots, n$. Hence,

$$\begin{aligned}
\delta L[\gamma] &= - \int_a^b \left\{ \frac{d}{dt} \left[\left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n 2g_{ij} \dot{q}^i \right] \delta q^j - \sum_{i,j,k=1}^n \frac{\partial g_{ij}}{\partial q^k} \delta q^k \dot{q}^i \dot{q}^j \right\} dt \\
&= - \int_a^b \left\{ \frac{d}{dt} \left[\left(\sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,k=1}^n g_{ik} \dot{q}^i \right] - \sum_{i,j,k=1}^n \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j \right\} \delta q^k dt.
\end{aligned}$$

The requirement that γ is a critical point of L with respect to all variations of γ vanishing at the endpoints implies that the integrand must vanish. Hence, we get a differential equations

$$\frac{d}{dt} \left[\left(\sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n g_{ik} \dot{q}^i \right] - \left(2 \sqrt{\sum_{i,j=1}^n g_{pq} \dot{q}^p \dot{q}^q} \right)^{-1} \sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j = 0. \quad (5)$$

This equation takes a very simple form if we assume that the curve γ is parametrized by its arc-length which we denote by

$$s(t) = \int_a^t \sqrt{\sum_{i,j=1}^n g_{ij}(q^1(t'), \dots, q^n(t')) \dot{q}^i(t') \dot{q}^j(t')} dt'.$$

Differentiating this equation with respect to s we get

$$\frac{ds}{dt} = \sqrt{\sum_{i,j=1}^n g_{ij}(q^1(t), \dots, q^n(t)) \dot{q}^i(t) \dot{q}^j(t)}.$$

In particular,

$$\sqrt{\sum_{i,j=1}^n g_{ij} \frac{dq^i}{ds} \frac{dq^j}{ds}} = 1.$$

Hence,

$$\frac{d}{ds} \left[\sum_{i=1}^n g_{ik} \frac{dq^i}{ds} \right] - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k} \frac{dq^i}{ds} \frac{dq^j}{ds} = 0. \quad (6)$$

Let (g^{ij}) be a symmetric matrix such that

$$\sum_{j=1}^n g^{ij} g_{jk} = \delta_k^j.$$

The collection (g^{ij}) gives components of a contravariant symmetric 2-tensor called the *contravariant metric tensor* associated to the metric g . We can rewrite equation (6) as follows.

$$\begin{aligned}
& \frac{d}{ds} \left[\sum_{i=1}^n g_{ik} \frac{dq^i}{ds} \right] - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k} \frac{dq^i}{ds} \frac{dq^j}{ds} \\
&= \sum_{i=1}^n g_{ik} \frac{d^2 q^i}{ds^2} + \sum_{i,j=1}^n \frac{\partial g_{ik}}{\partial x^j} \frac{dq^i}{ds} \frac{dq^j}{ds} - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k} \frac{dq^i}{ds} \frac{dq^j}{ds} \\
&= \sum_{i=1}^n g_{ik} \frac{d^2 q^i}{ds^2} + \frac{1}{2} \sum_{i,j=1}^n \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k} \right) \frac{dq^i}{ds} \frac{dq^j}{ds} = 0.
\end{aligned}$$

Contracting with g^{lk} we get

$$\frac{d^2 q^l}{ds^2} + \frac{1}{2} \sum_{i,j,k=1}^n g^{lk} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k} \right) \frac{dq^i}{ds} \frac{dq^j}{ds} = 0.$$

The quantities

$$\Gamma_{ij}^l = \frac{1}{2} \sum_{k=1}^n g^{lk} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial q^k} \right)$$

are called the *Christoffel symbols* (of the second type) corresponding to the metric g . It should be noted that the Christoffel symbols Γ_{ij}^l do not define a tensor on M . The equation of geodesics corresponding to the metric g are

$$\frac{d^2 q^l}{ds^2} + \sum_{i,j=1}^n \Gamma_{ij}^l \frac{dq^i}{ds} \frac{dq^j}{ds} = 0.$$

3 Linear connection.

We have no natural way of comparing vectors at different points of a manifold. Hence, if we want to write expressions involving derivatives of vector fields we need to introduce the notion of a linear connection on M .

Definition 7 *A linear connection on M is defined by a bilinear mapping*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$$

such that

$$\nabla_{fX} Y = f \nabla_X Y \text{ and } \nabla_X (fY) = (X(f))Y + f \nabla_X Y. \tag{7}$$

for every $f \in C^\infty(M)$.

The vector field $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X . The appearance of the derivative of f on the right hand side of equation (7) shows that ∇ is not a tensorial quantity.

Proposition 8 *The map*

$$\Theta : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

is an antisymmetric covariant 2-tensor on M .

Proof. Clearly, Θ is bilinear and antisymmetric. For every $f \in C^\infty(M)$,

$$\begin{aligned} \Theta(fX, Y) &= \nabla_{fX} Y - \nabla_Y fX - [fX, Y] = f\nabla_X Y - (Y(f))X - f\nabla_Y X - f[X, Y] + Y(f)X \\ &= f\Theta(X, Y). \end{aligned}$$

Hence, Θ is a covariant 2-tensor. ■

The 2-tensor Θ is called the torsion of the connection ∇ . A connection ∇ such that $\Theta = 0$ is called symmetric or torsion-free.

Given a linear connection ∇ on M we can define covariant derivatives of any tensorial quantities by requirement that the covariant derivative preserves the evaluation. In particular, if ω is a 1-form on M , then $\nabla_X \omega$ is a 1-form defined by the condition that

$$X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

for every $Y \in \mathfrak{X}(M)$. Hence,

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \quad \forall Y \in \mathfrak{X}(M).$$

Extensions to multi-tensors is given by multi-linearity. In particular, if g is a metric on M , then

$$(\nabla_X g)(Y_1, Y_2) = X(g(Y_1, Y_2)) - g(\nabla_X Y_1, Y_2) - g(Y_1, \nabla_X Y_2). \quad (8a)$$

In order to describe the linear connection ∇ in terms of a chart with coordinates q^1, \dots, q^n , it suffices to exhibit the covariant derivatives of the basic vector fields $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$. Let

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \sum_{k=1}^n C_{ij}^k \frac{\partial}{\partial q^k}.$$

The functions C_{ij}^k are called the connection coefficients. If $X = \sum_{i=1}^n X^i \frac{\partial}{\partial q^i}$ and $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial q^i}$ then

$$\begin{aligned}
\nabla_X Y &= \nabla_{\sum_{i=1}^n X^i \frac{\partial}{\partial q^i}} \left(\sum_{j=1}^n Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i=1}^n X^i \nabla_{\frac{\partial}{\partial q^i}} \left(\sum_{j=1}^n Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n X^i \nabla_{\frac{\partial}{\partial q^i}} \left(Y^j \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n \left(X^i \frac{\partial Y^j}{\partial q^i} \frac{\partial}{\partial q^j} + X^i Y^j \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} \right) \\
&= \sum_{i,j=1}^n \left(X^i \frac{\partial Y^j}{\partial q^i} \frac{\partial}{\partial q^j} + X^i Y^j \sum_{k=1}^n C_{ij}^k \frac{\partial}{\partial q^k} \right).
\end{aligned}$$

It follows that

$$\Theta(X, Y) = \sum_{i,j,k} (C_{ij}^k - C_{ji}^k) X^i Y^j \frac{\partial}{\partial q^k}.$$

In other words, the components of the torsion tensor are given by

$$\Theta_{ij}^k = C_{ij}^k - C_{ji}^k.$$

Suppose ∇ is a symmetric connection. That is $C_{ij}^k = C_{ji}^k$ in every coordinate system. We want to compute the covariant derivative of the metric tensor $g = \sum_{i,j=1}^n g_{ij} dq^i dq^j$. Equation (8a) with $X = \frac{\partial}{\partial q^k}$, $Y_1 = \frac{\partial}{\partial q^i}$ and $Y_2 = \frac{\partial}{\partial q^j}$ gives .

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial q^k}} g)_{ij} &= (\nabla_{\frac{\partial}{\partial q^k}} g) \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) \\
&= \frac{\partial}{\partial q^k} (g_{ij}) - g \left(\sum_{l=1}^n C_{ki}^l \frac{\partial}{\partial q^l}, \frac{\partial}{\partial q^j} \right) - g \left(\frac{\partial}{\partial q^i}, \sum_{l=1}^n C_{kj}^l \frac{\partial}{\partial q^l} \right) \\
&= \frac{\partial g_{ij}}{\partial q^k} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{kj}^l g_{il}.
\end{aligned}$$

Theorem 9 *Let g be a Riemannian metric on M . There exists a unique symmetric linear connection ∇ on M such that $\nabla g = 0$.*

Proof. $\nabla g = 0$ implies that

$$\frac{\partial g_{ij}}{\partial q^k} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{kj}^l g_{il} = 0.$$

Using the symmetry of C_{ij}^l , we can write

$$\begin{aligned}\frac{\partial g_{ik}}{\partial q^j} - \sum_{l=1}^n C_{ji}^l g_{lk} - \sum_{l=1}^n C_{kj}^l g_{il} &= 0, \\ \frac{\partial g_{kj}}{\partial q^i} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{ij}^l g_{kl} &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} &= \sum_{l=1}^n C_{ki}^l g_{lj} + \sum_{l=1}^n C_{ij}^l g_{kl} + \sum_{l=1}^n C_{ji}^l g_{lk} + \sum_{l=1}^n C_{kj}^l g_{il} - \sum_{l=1}^n C_{ki}^l g_{lj} - \sum_{l=1}^n C_{kj}^l g_{il} \\ &= 2 \sum_{l=1}^n C_{ij}^l g_{kl}.\end{aligned}$$

Contracting with g^{km} we get

$$C_{ij}^m = \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right) = \Gamma_{ij}^m.$$

■

Remark 10 *Remark that the connection coefficients given above are precisely the Christoffel symbols obtained when we considered the equation of geodesics. The connection defined here is called the Levi-Civita connection of the Riemannian metric g .*

Let ∇ be a linear connection on M and $\gamma : [a, b] \rightarrow M$ a curve in M . Consider a curve $\tilde{\gamma} : [a, b] \rightarrow TM$ such that $\tau(\tilde{\gamma}(t)) = \gamma(t)$. We say that the curve $\tilde{\gamma}$ in TM covers γ . For each $t \in [a, b]$, we have a vector $\tilde{\gamma}(t)$ attached $\gamma(t) \in M$. We can compute $\nabla_{\dot{\tilde{\gamma}}(t)} \tilde{\gamma}(t)$. In terms of coordinates q^1, \dots, q^n in M , $\gamma(t) = (q^1(t), \dots, q^n(t))$, and

$$\tilde{\gamma}(t) = \sum_{i=1}^n u^i(t) \frac{\partial}{\partial q^i} \text{ and } \dot{\tilde{\gamma}}(t) = \sum_{i=1}^n \dot{q}^i(t) \frac{\partial}{\partial q^i},$$

where $\dot{q}^i(t) = \frac{d}{dt} q^i(t)$. Then,

$$\begin{aligned}\nabla_{\dot{\tilde{\gamma}}(t)} \tilde{\gamma}(t) &= \nabla_{\left(\sum_{i=1}^n \dot{q}^i(t) \frac{\partial}{\partial q^i}\right)} \sum_{j=1}^n u^j(t) \frac{\partial}{\partial q^j} \\ &= \sum_{j=1}^n \frac{du^j(t)}{dt} \frac{\partial}{\partial q^j} + \sum_{i,j=1}^n \dot{q}^i u^j \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} \\ &= \sum_{j=1}^n \frac{du^j(t)}{dt} \frac{\partial}{\partial q^j} + \sum_{i,j,k=1}^n \dot{q}^i u^j \Gamma_{ij}^k \frac{\partial}{\partial q^k}.\end{aligned}$$

Definition 11 If $\nabla_{\dot{\gamma}(t)}\tilde{\gamma}(t) = 0$ for all $t \in [a, b]$, we say that the curve $\tilde{\gamma}(t)$ gives the parallel transport along the curve $t \mapsto \gamma(t)$ of the vector $\tilde{\gamma}(a) \in T_{\gamma(a)}M$.

Consider now the equation of geodesics $\frac{d^2q^l}{ds^2} + \sum_{i,j=1}^n \Gamma_{ij}^l \frac{dq^i}{ds} \frac{dq^j}{ds} = 0$. In this case we take $\tilde{\gamma} = \dot{\gamma}$, and parametrize the curve γ by arc-length s . Therefore,

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(s) = \sum_{j=1}^n \frac{d\dot{q}(s)}{ds} \frac{\partial}{\partial \dot{q}^j} + \sum_{i,j,k=1}^n \dot{q}^i \dot{q}^j \Gamma_{ij}^k \frac{\partial}{\partial \dot{q}^k}.$$

Therefore the equation of geodesics is equivalent to

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(s) = 0.$$

In other words, a curve $t \mapsto \gamma(t)$ is a geodesics if its tangent vector $\dot{\gamma}(t)$ is parallelly transported along γ .

4 Curvature

The operators ∇_X and ∇_Y need not commute even if the vector fields X and Y commute. Consider the map

$$R : (\mathfrak{X}(M))^3 \rightarrow \mathfrak{X}(M) : (X, Y, Z) \mapsto R(X, Y, Z) = \nabla_X(\nabla_Y(Z)) - \nabla_Y(\nabla_X(Z)) - \nabla_{[X,Y]}Z. \quad (9)$$

Proposition 12 R is a covariant 3-tensor field with values in $\mathfrak{X}(M)$. In other words, R is a mixed tensor field of type $(3, 1)$.

Proof. We need to show that, for each X, Y, Z in $\mathfrak{X}(M)$ and $f_1, f_2, f_3 \in C^\infty(M)$,

$$\nabla_{f_1 X}(\nabla_{f_2 Y}(f_3 Z)) - \nabla_{f_2 Y}(\nabla_{f_1 X}(f_3 Z)) - \nabla_{[f_1 X, f_2 Y]}f_3 Z = f_1 f_2 f_3 (\nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - [X, Y]Z).$$

We consider each factor separately.

$$\begin{aligned} & \nabla_{f_1 X}(\nabla_Y(Z)) - \nabla_Y(\nabla_{f_1 X}(Z)) - \nabla_{[f_1 X, Y]}Z \\ &= f_1 \nabla_X(\nabla_Y(Z)) - \nabla_Y(f_1 \nabla_X(Z)) - \nabla_{f_1[X, Y] - Y(f_1)X}Z \\ &= f_1 \nabla_X(\nabla_Y(Z)) - f_1 \nabla_Y(\nabla_X(Z)) - Y(f_1) \nabla_X Z - f_1 \nabla_{[X, Y]} + Y(f_1) \nabla_X Z \\ & \quad f_1 (\nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - [X, Y]Z). \end{aligned}$$

By symmetry we have

$$\nabla_X(\nabla_{f_2 Y}(Z)) - \nabla_{f_2 Y}(\nabla_X(Z)) - \nabla_{[X, f_2 Y]}Z = f_2 (\nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - [X, Y]Z).$$

Finally

$$\begin{aligned}
& \nabla_X(\nabla_Y(f_3Z)) - \nabla_Y(\nabla_X(f_3Z)) - \nabla_{[X,Y]}f_3Z \\
= & \nabla_X(f_3\nabla_Y(Z) + Y(f_3)Z) - \nabla_Y(f_3\nabla_X(Z) + X(f_3)Z) - f_3\nabla_{[X,Y]}Z - [X,Y](f_3)Z \\
= & f_3\nabla_X(\nabla_Y(Z)) + X(f_3)\nabla_YZ + Y(f_3)\nabla_XZ + X(Y(f_3)) - f_3\nabla_Y(\nabla_X(Z)) - Y(f_3)\nabla_XZ + \\
& -Y(X(f_3))Z - X(f_3)\nabla_YZ - f_3\nabla_{[X,Y]}Z - [X,Y](f_3)Z \\
= & f_3(\nabla_Y(\nabla_XZ) - \nabla_X(\nabla_YZ)) - [X,Y]Z.
\end{aligned}$$

■

Definition 13 *The tensor field R defined in equation (9) is called the Riemann tensor of the Levi-Civita connection of the metric g .*

A Riemannian manifold (M, g) of dimension n is said to be flat if there is a diffeomorphism φ of M onto an open subset of \mathbb{R}^n such that

$$\varphi^*g_{\mathbb{R}^n} = g. \quad (10)$$

Equation (10) mean that

$$g(u, v) = g_{\mathbb{R}^n}(T\varphi(u), T\varphi(v))$$

for every $x \in M$ and $u, v \in T_xM$.

We say that (M, g) is *locally flat* if, for every $x \in M$ there is a neighbourhood U of x such that the restriction $g|_U$ of the metric g to U is flat. Thus, (M, g) is locally flat if, for each $x \in M$, there is a chart $\varphi : U \rightarrow V$, where U is a neighbourhood of x in M and V is an open set in \mathbb{R}^n , such that the components of the metric in this chart are

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Theorem 14 *A Riemannian manifold is locally flat if and only if the Riemann tensor vanishes identically.*

Proof. See Mike Spivak, *Differential Geometry*, vol 2, Chapter 4, Theorem 13. ■

Problem 15 *Find the Riemann tensor of the 2-sphere of radius r*

$$S_r^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$$

with metric g obtained as the pull-back of the Euclidean metric $g_{\mathbb{R}^3}$ by the inclusion map $S^2 \hookrightarrow \mathbb{R}^3$.