1. The solution of $y'' + y = 0$ is of course $y = c_1 \sin x + c_2 \cos x$. As series, we know (letting $0! = 1$) that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_0^\infty \frac{x^{2k+1}}{(2k+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_0^\infty \frac{x^{2k}}{(2k)!}.$$

The same applies to $y'' - y = 0$ with solution $y = c_1 \sinh x + c_2 \cosh x$.

(a) Find the Taylor expansions of $\sinh x$ and $\cosh x$.

(b) With $i = \sqrt{-1}$, let

$$f(x) = \sin(ix) = \sum_0^\infty \frac{(ix)^{2k+1}}{(2k+1)!}.$$

Express f'' in terms of f and relate it to your answer in (a).

(c) Repeat the previous question for

$$g(x) = \cos(ix) = \sum_0^\infty \frac{(ix)^{2k}}{(2k)!}.$$

(d) In a cylindrical problem involving Laplace's equation separation of variables gave the equation for $R(r)$ as

$$R'' + \frac{1}{r}R' - \lambda^2 R = 0 \qquad (*)$$

(which would be Bessel's equation of order zero if it was $+\lambda^2$.) Make the change of variable $t = i\lambda r$ to get

$$\frac{d^2R}{dt^2} + \frac{1}{t}\frac{dR}{dt} + R = 0,$$

which we recognize as Bessel's equation of order zero in the variable t. So up to constant multiples, the only solution of $(*)$ which is continuous at the origin $r = 0$ is

$$R(r) = J_0(i\lambda r).$$

Use the series expansion for J_0 to get

$$J_0(i\lambda r) = \sum_0^\infty \frac{(\lambda r)^{2k}}{2^{2k}(k!)^2},$$

so that $J_0(i\lambda r)$ is a real valued function after all.
In view of this, we adopt a notation which does not (misleadingly) involve i, and so we set

$$I_0(x) = \sum_0^\infty \frac{x^{2k}}{2^{2k}(k!)^2}.$$

I_0 is called the *modified Bessel function of order zero of the first kind*, and is related to the Bessel function of order zero in the same way as $\sin x$ and $\cos x$ are related to $\sinh x$ and $\cosh x$.