

1. Compute the Fourier coefficients of the function

$$f(x) = x^2 \quad -\pi \leq x \leq \pi.$$

2. Find the Fourier series of $f(x) = \sin^3(x)$.
3. If a_k for $k = 0, 1, 2, \dots$ and b_k for $k = 1, 2, \dots$ are the Fourier coefficients of $f(x)$, show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This is called Parseval's equality.

4. Let $w = (x^2 - 1)^n$, and denote the k th derivative of w by $w^{(k)}$.
 - (a) Show that

$$\int_{-1}^1 w^{(n)} w^{(n)} dx = (2n)! \int_{-1}^1 (1-x)^n (1+x)^n dx.$$

- (b) Prove that

$$\int_{-1}^1 (1-x)^n (1+x)^n dx = \frac{(n!)^2}{(2n)!(2n+1)} 2^{2n+1}$$

- (c) Use the previous results to show that

$$\int_{-1}^1 P_n(x)^2 dx = \frac{2}{2n+1}.$$

Here P_n is the n th Legendre polynomial.

5. Let $\alpha > -1$ be a real number and let

$$L_n^{(\alpha)}(x) = (-1)^n x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}),$$

for $n = 0, 1, 2, 3, \dots$. Show that $L_n^{(\alpha)}(x)$ is a polynomial in x of degree n with leading coefficient 1 for all α and all n .

6. Find all eigenvalues and eigenvectors for the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

7. (a) Show that the boundary-value problem consisting of the fourth-order differential equation

$$\frac{d^4 y}{dx^4} - \omega^2 y = 0$$

and boundary conditions

$$y(0) = y(1) = 0, \quad y'(0) = y'(1) = 0$$

had nontrivial solutions if and only if

$$\cos \sqrt{\omega} = \frac{1}{\cosh \sqrt{\omega}}.$$

- (b) Show that the boundary-value problem has infinitely many non-negative eigenvalues ω_n , $n = 0, 1, 2, \dots$. How do these eigenvalues behave as $n \rightarrow \infty$?
- (c) What is the general solution of the boundary-value problem corresponding to the eigenvalue ω_n ?