1. Find the general solution of the Euler equation
$$x^2y'' + xy' - p^2y = 0,$$
p a constant.

2. Find a function $u(x, y)$ constant on all lines parallel to $3x + 2y = 0$, and such that $u(x, 0) = x^2 + 2x + 1$.

3. Solve the wave equation $u_{xx} = u_{tt}$ (so the wave speed is 1) on the line if $u(x, 0) = \sin x$ and $u_t(x, 0) = \cos x$.

4. Find the Fourier series for $f(x) = |\sin x|$ on $-\pi < x < \pi$.

5. Recall that the Hermite polynomial H_n of degree n is a solution of the second order differential equation
$$y'' - xy' + ny = 0.$$
Show that the Hermite functions $h_n(x) = e^{-x^2/4}H_n(x)$ satisfy the linear differential equation
$$y'' + \left(n + \frac{1}{2} - \frac{x^2}{4}\right) y = 0.$$

6. Prove that every linear transformation on an odd-dimensional vector space has at least one real eigenvalue. Give a geometric interpretation of this result for linear transformations on three space.

7. On the Euclidean space of polynomials with
$$p \cdot q = \int_{-1}^{1} p(x)q(x)\, dx$$
is the linear transformation L given by $Lp(x) = p'(x)$ a symmetric transformation?

8. Find all solutions of the boundary value problem $y''+9y = 0$; $y'(0) = 0$, $y'(\pi) = 0$. (I get $y = c\cos 3x$).

9. Show that the solution $u(x,t)$ of the one-dimensional wave equation $u_{tt} = a^2 u_{xx}$ on the interval $[0,\pi]$ subject to the endpoint conditions $u(0,t) = u(\pi,t) = 0$ and initial conditions $u(x,0) = x(\pi - x)$, $u_t(x,0) = 0$ is
$$u(x,t) = \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{n^3}[1 + (-1)^{n+1}] \cos nat \sin nx$$

10. The temperature in a slender insulated rod of length L satisfies the endpoint conditions $u(0,t) = 0$, $u(L,t) = 1$, and initial condition $u(x,0) = \sin(\pi x/L)$. What is the *steady state* temperature in the rod (i.e. the temperature as $t \to \infty$)?

11. Show that the Frobenius method applied to the equation
$$8x^2 y'' - 2x(x-1)y' + (x+1)y = 0$$
gives the solutions
$$y_1(x) = x^{1/2}, \qquad y_2(x) = x^{1/4} \sum_{k=0}^{\infty} \frac{1}{k! 2^{2k}(4k-1)} x^k.$$

12. Recall that the Bessel function $J_n(x)$ has the series expansion
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n}.$$
Use this to show that the derivative
$$\frac{d}{dx} J_0(x) = -J_1(x).$$