

Computing modular polynomials with the Chinese Remainder Theorem

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Isogenies

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Over a finite field, E_1 and E_2 are isogenous if and only if

$$\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q).$$

Some applications of isogenies

Isogenies make hard problems easier:

- ▶ Counting the points on E
Polynomial time (SEA).
- ▶ Constructing E with the CM method.
 $|D| \geq 10^{14}$ $h(D) \geq 5,000,000$ (CRT approach).
- ▶ Computing the endomorphism ring of E .
Subexponential time (heuristically, S-Bisson 2009).

These algorithms all rely on modular polynomials $\Phi_\ell(X, Y)$.

Isogenies in elliptic curve cryptography

Isogenies allow the discrete logarithm problem to be transferred from one elliptic curve to another.

This raises a few questions:

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2. Are all isogenous curves created equal?

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The endomorphism ring $\text{End}(E)$ is critical to both questions.

(Bröker-Charles-Lauter 2008, Jao-Miller-Venkatesan 2005, 2009).

Properties of isogenies

Degree

The kernel of $\phi : E_1 \rightarrow E_2$ is a finite subgroup of $E_1(\overline{F})$.
When ϕ is separable, we have $|\ker \phi| = \deg \phi$.

An ℓ -isogeny is a (separable) isogeny of degree ℓ .
For prime ℓ , the kernel is necessarily cyclic.

Orientation

We say that $\phi : E_1 \rightarrow E_2$ is *horizontal* if $\text{End}(E_1) = \text{End}(E_2)$.
Otherwise ϕ is *vertical*.

CM-action

Let E/\mathbb{F}_q be an ordinary elliptic curve.

Then $\text{End}(E) \cong \mathcal{O} \subseteq \mathcal{O}_K$, for some imaginary quadratic field K .

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Under the ERH this is always true, and “small” = $O(\log^2 |D|)$.

Isogenies from kernels

Any finite subgroup G of $E(\overline{\mathbb{F}})$ determines a separable isogeny with G as its kernel

Given G , we can compute ϕ explicitly via Vélu's formula.

The complexity depends both on the size of $\ker \phi$, and the field in which the points of $\ker \phi$ are defined.

When working in \mathbb{F}_q , we assume the coefficients of ϕ lie in \mathbb{F}_q . But $\ker \phi$ may lie in an extension of degree up to $\ell^2 - 1$.

The classical modular polynomial Φ_ℓ

The symmetric polynomial $\Phi_\ell \in \mathbb{Z}[X, Y]$ has the property

$$\Phi_\ell(j(E_1), j(E_2)) = 0 \iff E_1 \text{ and } E_2 \text{ are } \ell\text{-isogenous.}$$

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The ℓ -isogeny graph has vertex set $\{j(E) : E/\mathbb{F}_q\}$
and edges (j_1, j_2) whenever $\Phi_\ell(j_1, j_2) = 0$ (in \mathbb{F}_q).

The neighbors of j are the roots of $\Phi_\ell(X, j) \in \mathbb{F}_q[X]$.

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Φ_ℓ is big: $O(\ell^3 \log \ell)$ bits.

This is a pretty big polynomial...

$H_D(x)$



Visible
Universe

...but this is a *really* big polynomial.

$$\Phi_l(x, y)$$

$$H_D(x)$$



| ℓ | coefficients | largest | average | total |
|--------|--------------|---------|---------|--------|
| 127 | 8258 | 7.5kb | 5.3kb | 5.5MB |
| 251 | 31880 | 16kb | 12kb | 48MB |
| 503 | 127262 | 36kb | 27kb | 431MB |
| 1009 | 510557 | 78kb | 60kb | 3.9GB |
| 2003 | 2009012 | 166kb | 132kb | 33GB |
| 3001 | 4507505 | 259kb | 208kb | 117GB |
| 4001 | 8010005 | 356kb | 287kb | 287GB |
| 5003 | 12522512 | 454kb | 369kb | 577GB |
| 10007 | 50085038 | 968kb | 774kb | 4.8TB* |

Size of $\Phi_\ell(X, Y)$

*Estimated

Algorithms to compute Φ_ℓ

q-expansions:

(Atkin ?, Elkies '92, '98, LMMS '94, Morain '95, Müller '95, BCRS '99)

$$\Phi_\ell: \quad O(\ell^4 \log^{3+\epsilon} \ell) \quad (\text{via the CRT})$$

$$\Phi_\ell \bmod p: \quad O(\ell^3 \log \ell \log^{1+\epsilon} p) \quad (p > \ell + 1)$$

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evaluation-interpolation: (Enge 2009)

$$\Phi_\ell: \quad O(\ell^3 \log^{4+\epsilon} \ell) \quad (\text{floating-point})$$

$$\Phi_\ell \bmod m: \quad O(\ell^3 \log^{4+\epsilon} \ell) \quad (\text{reduces } \Phi_\ell)$$

A new algorithm to compute Φ_ℓ

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In practice the algorithm is much faster than other methods. It is probabilistic, but the output is unconditionally correct.

Performance highlights

Level records

1. $\ell = 5003$: Φ_ℓ
2. $\ell = 10007$: $\Phi_\ell \bmod m$
3. $\ell = 50021$: Φ_ℓ^f

Each in less than 24 hours elapsed time (≈ 12 CPU-days), using $m \approx 2^{256}$.

Speed records

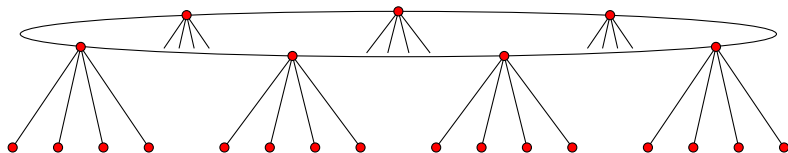
1. $\ell = 251$: Φ_ℓ in 40s $\Phi_\ell \bmod m$ in 5.5s
2. $\ell = 1009$: Φ_ℓ in 3822s $\Phi_\ell \bmod m$ in 408s
3. $\ell = 1009$: Φ_ℓ^f in 3.2s

Single core CPU times (AMD 3.0 GHz), using $m \approx 2^{256}$.

Effective throughput when computing $\Phi_{1009} \bmod m$ is 100Mb/s.



Mapping a volcano



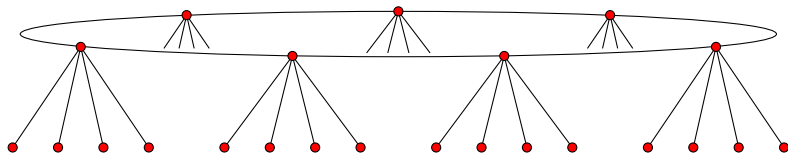
Mapping a volcano

Example

$$\ell = 5, \quad p = 4451, \quad D = -151$$

General requirements

$$4p = t^2 - v^2 \ell^2 D, \quad p \equiv 1 \pmod{\ell}$$



Mapping a volcano

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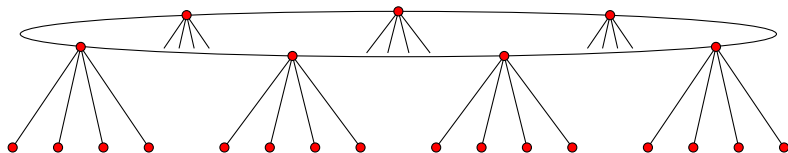
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$$\ell \nmid v, \quad \left(\frac{D}{\ell}\right) = 1, \quad h(D) \geq \ell + 2$$



Mapping a volcano

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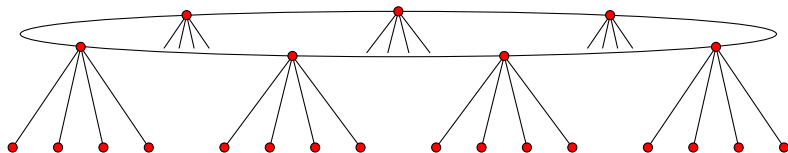
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1. Find a root of $H_D(X)$

Mapping a volcano

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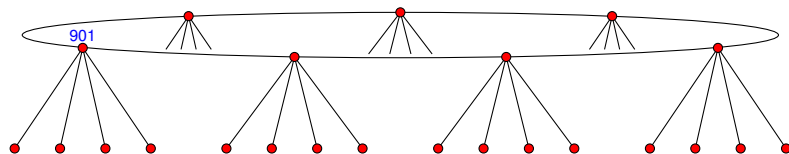
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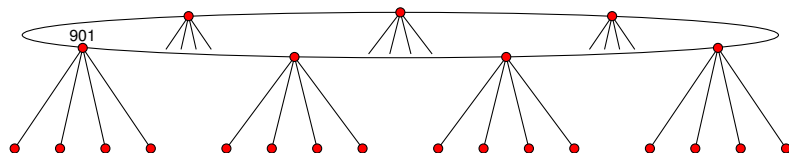
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2. Enumerate surface using the action of α_{ℓ_0}

Mapping a volcano

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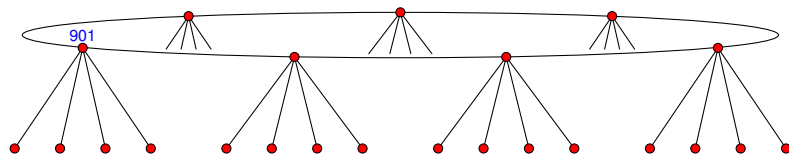
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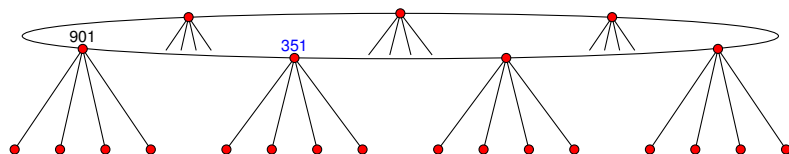
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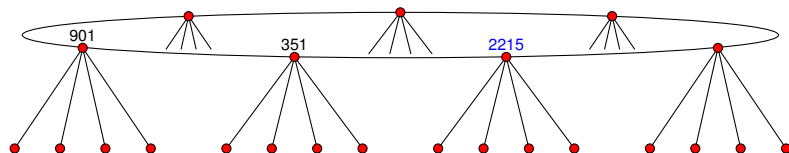
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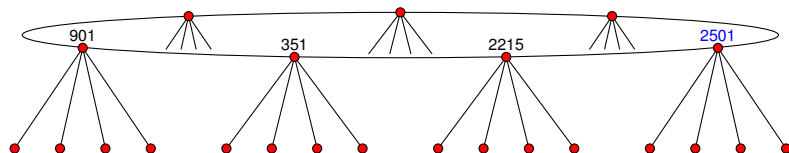
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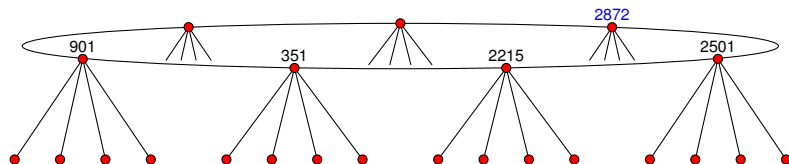
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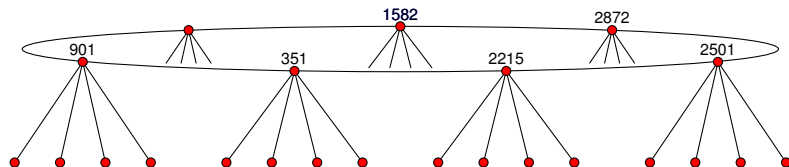
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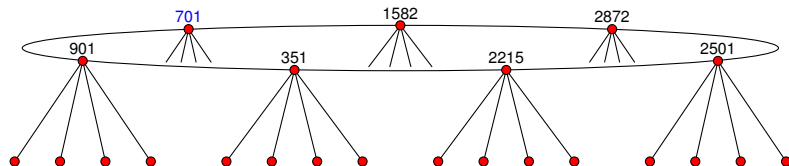
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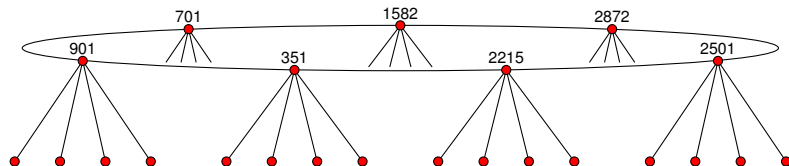
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Mapping a volcano

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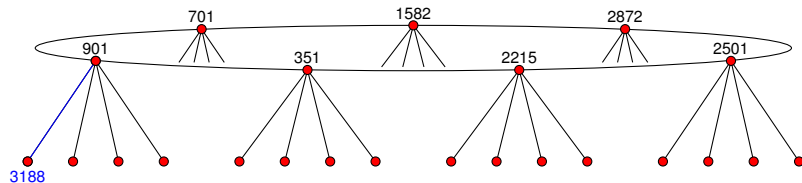
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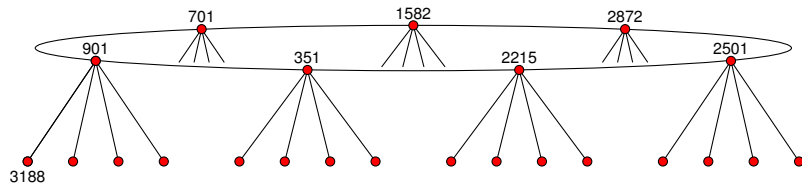
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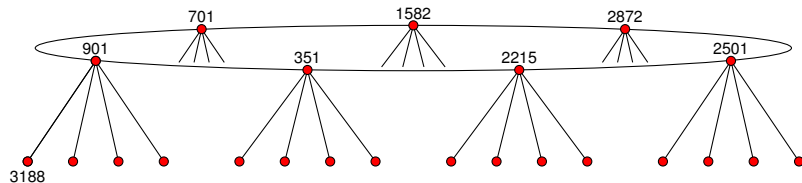
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$$\begin{array}{cccccccccccc}
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 \end{array}$$

Mapping a volcano

Example

$$\ell = 5, \quad p = 4451, \quad D = -151$$

$$t = 52, \quad v = 2, \quad h(D) = 7$$

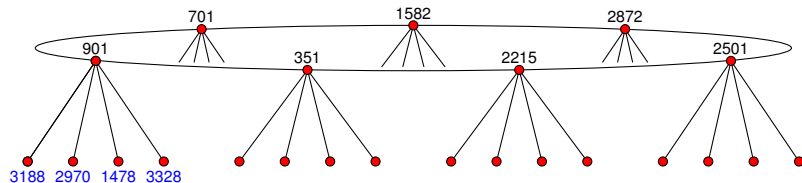
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$$4p = t^2 - v^2 \ell^2 D, \quad p \equiv 1 \pmod{\ell}$$

$$\ell \nmid v, \quad \left(\frac{D}{\ell}\right) = 1, \quad h(D) \geq \ell + 2$$

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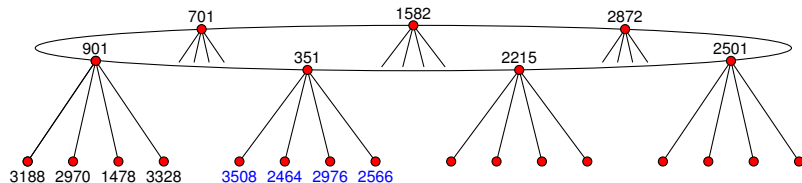
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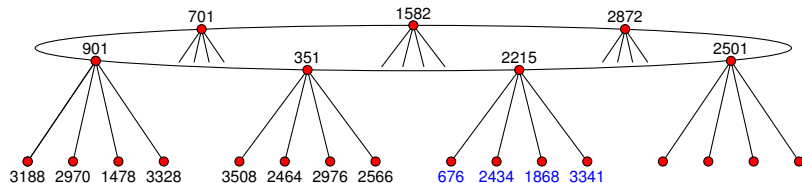
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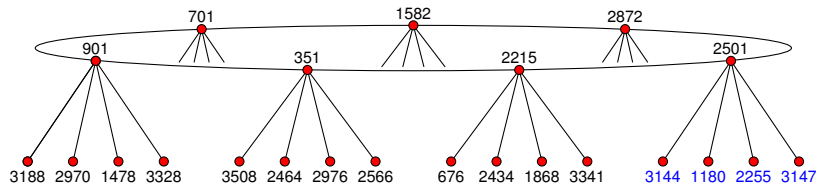
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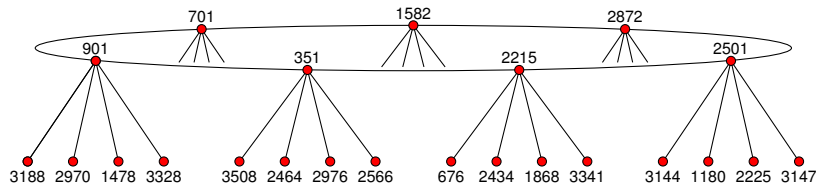
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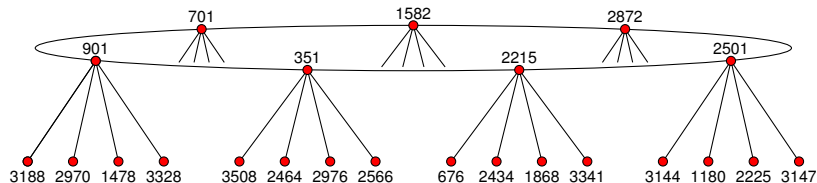
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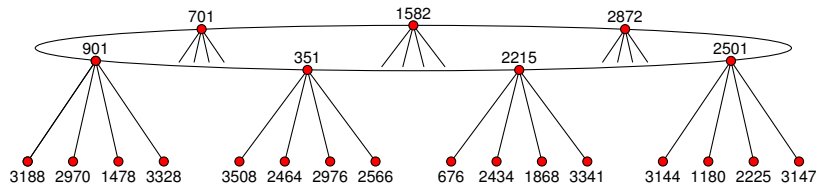
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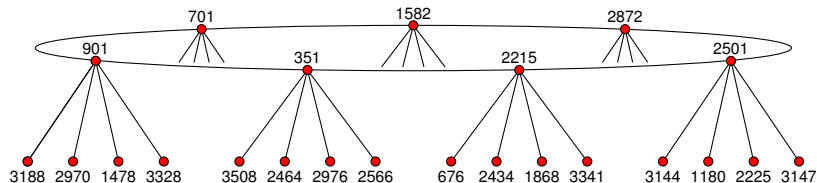
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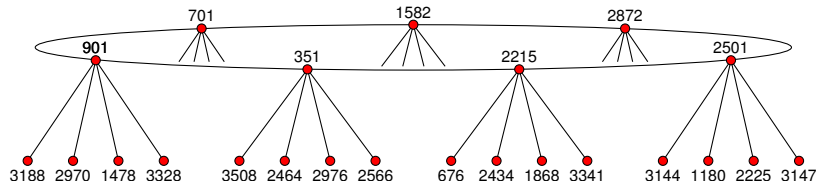
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Interpolation



$$\Phi_5(X, 901) = (X - 701)(X - 351)(X - 3188)(X - 2970)(X - 1478)(X - 3328)$$

$$\Phi_5(X, 351) = (X - 901)(X - 2215)(X - 3508)(X - 2464)(X - 2976)(X - 2566)$$

$$\Phi_5(X, 2215) = (X - 351)(X - 2501)(X - 3341)(X - 1868)(X - 2434)(X - 676)$$

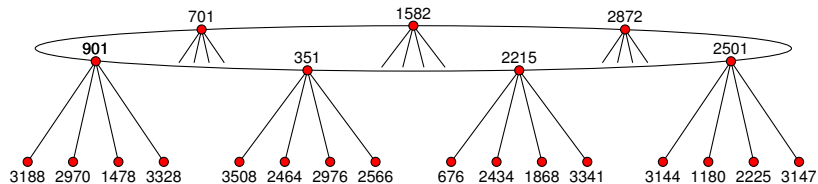
$$\Phi_5(X, 2501) = (X - 2215)(X - 2872)(X - 3147)(X - 2255)(X - 1180)(X - 3144)$$

$$\Phi_5(X, 2872) = (X - 2501)(X - 1582)(X - 1502)(X - 4228)(X - 1064)(X - 2087)$$

$$\Phi_5(X, 1582) = (X - 2872)(X - 701)(X - 945)(X - 3497)(X - 3244)(X - 291)$$

$$\Phi_5(X, 701) = (X - 1582)(X - 901)(X - 2843)(X - 4221)(X - 3345)(X - 4397)$$

Interpolation



$$\Phi_5(X, 901) = X^6 + 1337X^5 + 543X^4 + 497X^3 + 4391X^2 + 3144X + 3262$$

$$\Phi_5(X, 351) = X^6 + 3174X^5 + 1789X^4 + 3373X^3 + 3972X^2 + 2932X + 4019$$

$$\Phi_5(X, 2215) = X^6 + 2182X^5 + 512X^4 + 435X^3 + 2844X^2 + 2084X + 2709$$

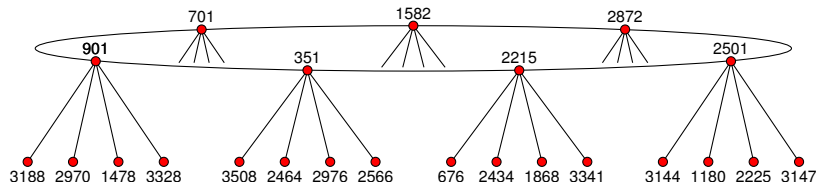
$$\Phi_5(X, 2501) = X^6 + 2991X^5 + 3075X^4 + 3918X^3 + 2241X^2 + 3755X + 1157$$

$$\Phi_5(X, 2872) = X^6 + 389X^5 + 3292X^4 + 3909X^3 + 161X^2 + 1003X + 2091$$

$$\Phi_5(X, 1582) = X^6 + 1803X^5 + 794X^4 + 3584X^3 + 225X^2 + 1530X + 1975$$

$$\Phi_5(X, 701) = X^6 + 515X^5 + 1419X^4 + 941X^3 + 4145X^2 + 2722X + 2754$$

Interpolation



$$\begin{aligned}
 \Phi_5(X, Y) = & X^6 + (4450Y^5 + 3720Y^4 + 2433Y^3 + 3499Y^2 + 70Y + 3927)X^5 \\
 & (3720Y^5 + 3683Y^4 + 2348Y^3 + 2808Y^2 + 3745Y + 233)X^4 \\
 & (2433Y^5 + 2348Y^4 + 2028Y^3 + 2025Y^2 + 4006Y + 2211)X^3 \\
 & (3499Y^5 + 2808Y^4 + 2025Y^3 + 4378Y^2 + 3886Y + 2050)X^2 \\
 & (70Y^5 + 3745Y^4 + 4006Y^3 + 3886Y^2 + 905Y + 2091)X \\
 & (Y^6 + 3927Y^5 + 233Y^4 + 2211Y^3 + 2050Y^2 + 2091Y + 2108)
 \end{aligned}$$

Computing $\Phi_\ell(X, Y) \bmod p$

Assume D and p are suitably chosen with $D = O(\ell^2)$ and $\log p = O(\log \ell)$, and that $H_D(X)$ has been precomputed.

1. Find a root of $H_D(X)$ over \mathbb{F}_p . $O(\ell \log^{3+\epsilon} \ell)$
2. Enumerate the surface(s) using $\text{cl}(D)$ -action. $O(\ell \log^{2+\epsilon} \ell)$
3. Descend to the floor using Vélú. $O(\ell \log^{1+\epsilon} \ell)$
4. Enumerate the floor using $\text{cl}(\ell^2 D)$ -action. $O(\ell^2 \log^{2+\epsilon} \ell)$
5. Build each $\Phi_\ell(X, j_i)$ from its roots. $O(\ell^2 \log^{3+\epsilon} \ell)$
6. Interpolate $\Phi_\ell(X, Y) \bmod p$. $O(\ell^2 \log^{3+\epsilon} \ell)$

Time complexity is $O(\ell^2 \log^{3+\epsilon} \ell)$.

Space complexity is $O(\ell^2 \log \ell)$.

After computing $\Phi_5(X, Y) \bmod p$ for the primes:

4451, 6911, 9551, 28111, 54851, 110051, 123491, 160591, 211711, 280451, 434111, 530851, 686051, 736511,

we apply the CRT to obtain

$$\begin{aligned}\Phi_5(X, Y) = & X^6 + Y^6 - X^5 Y^5 + 3720(X^5 Y^4 + X^4 Y^5) - 4550940(X^5 Y^3 + X^3 Y^5) \\ & + 2028551200(X^5 Y^2 + X^2 Y^5) - 246683410950(X^5 Y + X Y^5) + 1963211489280(X^5 + Y^5) \\ & + 1665999364600X^4 Y^4 + 107878928185336800(X^4 Y^3 + X^3 Y^4) \\ & + 383083609779811215375(X^4 Y^2 + X^2 Y^4) + 128541798906828816384000(X^4 Y + X Y^4) \\ & + 1284733132841424456253440(X^4 + Y^4) - 4550940(X^3 Y^5 + X^5 Y^3) \\ & - 441206965512914835246100X^3 Y^3 + 26898488858380731577417728000(X^3 Y^2 + X^2 Y^3) \\ & - 192457934618928299655108231168000(X^3 Y + X Y^3) \\ & + 280244777828439527804321565297868800(X^3 + Y^3) \\ & + 5110941777552418083110765199360000X^2 Y^2 \\ & + 36554736583949629295706472332656640000(X^2 Y + X Y^2) \\ & + 6692500042627997708487149415015068467200(X^2 + Y^2) \\ & - 264073457076620596259715790247978782949376XY \\ & + 53274330803424425450420160273356509151232000(X + Y) \\ & + 141359947154721358697753474691071362751004672000.\end{aligned}$$

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(but note that $\Phi_5^f(X, Y) = X^6 + Y^6 - X^5 Y^5 + 4XY$).

The algorithm

Given a prime $\ell > 2$ and an integer $m > 0$:

1. Pick a discriminant D suitable for ℓ .
2. Select a set of primes S suitable for ℓ and D .
3. Precompute H_D , $\text{cl}(D)$, $\text{cl}(\ell^2 D)$, and CRT data.
4. For each $p \in S$, compute $\Phi_\ell \bmod p$ and update CRT data.
5. Perform CRT postcomputation and output $\Phi_\ell \bmod m$.

To compute Φ_ℓ over \mathbb{Z} , just use $m = \prod p$.

For “small” m , use explicit CRT mod m .

For “large” m , standard CRT for large m .

For m in between, use a hybrid approach.

Chinese remaindering

Let $S = \{p_1, \dots, p_n\}$, $M = \prod p_i$, $M_i = M/p_i$, and $a_i \equiv M_i^{-1} \pmod{p_i}$.
For each coefficient c of Φ_ℓ , let $c_i \equiv c \pmod{p_i}$ and assume $4|c| < M$.

Standard CRT: $c \equiv \sum c_i a_i M_i \pmod{M}$.

Explicit CRT mod m [Bernstein]:

$$c \equiv \left(\sum c_i a_i M_i - rM \right) \pmod{m}$$

where r is the closest integer to $\sum c_i a_i / M_i$.

Online algorithm: process each c_i as it is computed, then discard it!

Assuming $\log p_i = O(\log l)$:

- ▶ Space complexity: $O(\ell^2 \log(\ell m))$.
- ▶ Time complexity: $O(\ell^3 \log^{3+\epsilon} \ell + \ell^3 M(\log m))$

With hybrid approach, time is $O(\ell^3 \log^{3+\epsilon} \ell)$ independent of m .

See [arXiv.org/abs/0902.4670](https://arxiv.org/abs/0902.4670) for more details.

Complexity

Theorem (GRH)

For every prime $\ell > 2$ there is a suitable discriminant D with $|D| = O(\ell^2)$ for which there are $\Omega(\ell^3 \log^3 \ell)$ primes $p = O(\ell^6 (\log \ell)^4)$ that are suitable for ℓ and D .

Heuristically, $p = O(\ell^4)$. In practice, $\lg p < 64$.

Theorem (GRH)

*The expected running time is $O(\ell^3 \log^3 \ell \log \log \ell)$.
The space required is $O(\ell^2 \log(\ell m))$.*

An explicit height bound for Φ_ℓ

Let ℓ be a prime.

Let $h(\Phi_\ell)$ be the (natural) logarithmic height of Φ_ℓ .

Asymptotic bound: $h(\Phi_\ell) = 6\ell \log \ell + O(\ell)$ (Paula Cohen 1984).

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Explicit bound: $h(\Phi_\ell) \leq 6\ell \log \ell + 17\ell$ (Bröker-S 2009).

Conjectural bound: $h(\Phi_\ell) \leq 6\ell \log \ell + 12\ell$ (for $\ell > 30$).

The explicit bound holds for all ℓ .

The conjectural bound is known to hold for $30 < \ell < 3600$.

Other modular functions

We can compute polynomials relating $f(z)$ and $f(\ell z)$ for other modular functions, including the Weber f -function.

The coefficients of Φ_ℓ^f are roughly 72 times smaller.
This means we need 72 fewer primes.

The polynomial Φ_ℓ^f is roughly 24 times sparser.
This means we need 24 times fewer interpolation points.

Overall, we get nearly a 1728-fold speedup using Φ_ℓ^f .

Modular polynomials for $\ell = 11$

Classical:

$$\begin{aligned} X^{12} + Y^{12} - X^{11}Y^{11} + -1X^{11}Y^{11} + 8184X^{11}Y^{10} - 28278756X^{11}Y^9 + 53686822816X^{11}Y^8 \\ - 61058988656490X^{11}Y^7 + 42570393135641712X^{11}Y^6 - 17899526272883039048X^{11}Y^5 \\ + 4297837238774928467520X^{11}Y^4 - 529134841844639613861795X^{11}Y^3 + 27209811658056645815522600X^{11}Y^2 \\ - 374642006356701393515817612X^{11}Y + 296470902355240575283200000X^{11} \\ \dots 8 \text{ pages omitted} \dots \\ + 392423345094527654908696 \dots 100 \text{ digits omitted} \dots 000 \end{aligned}$$

Atkin:

$$\begin{aligned} X^{12} - X^{11}Y + 744X^{11} + 196680X^{10} + 187X^9Y + 21354080X^9 + 506X^8Y + 830467440X^8 \\ - 11440X^7Y + 16875327744X^7 - 57442X^6Y + 208564958976X^6 + 184184X^5Y + 1678582287360X^5 \\ + 1675784X^4Y + 9031525113600X^4 + 1867712X^3Y + 32349979904000X^3 - 8252640X^2Y + 74246810880000X^2 \\ - 19849600XY + 98997734400000X + Y^2 - 8720000Y + 58411072000000 \end{aligned}$$

Weber:

$$X^{12} + Y^{12} - X^{11}Y^{11} + 11X^9Y^9 - 44X^7Y^7 + 88X^5Y^5 - 88X^3Y^3 + 32XY$$

Performance comparison

| function | ℓ | floating-point | CRT | ratio |
|---------------|--------|----------------|------|-------|
| classical j | 251 | 688 | 40 | 17.2 |
| | 503 | 8320 | 410 | 20.3 |
| | 1009 | 107200 | 3822 | 28.0 |
| Weber f | 1009 | 16.2 | 3.2 | 5.1 |
| | 5003 | 4504 | 492 | 9.2 |
| | 10007 | 66758 | 4931 | 13.5 |

floating-point vs. CRT

(3.0 GHz AMD Phenom CPU seconds, $m \approx 2^{256}$)