Stable Modules and a Theorem of Camillo and Yu

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June 19, 2013

Abstract

In 1995, Camillo and Yu showed that an exchange ring has stable range 1 if and only if every regular element is unit regular. An element m in a module ${}_RM$ is called regular if $(m\lambda)m = m$ for some $\lambda \in \hom(M, R)$. In this paper we define stable modules and show that if M has the finite exchange property then M is stable if and only if, for every regular element $m \in M$, $(m\gamma)m = m$ where $\gamma : M \to R$ is epic (and we say that m is unit regular). Such modules are called regular stable. It is shown that ${}_RR$ is regular-stable if and only if R has internal cancellation. To simplify the exposition, many arguments are formulated in an arbitrary Morita context.

Key Words: Regular module; Stable module; Regular-stable module; Stable range one; Morita contexts.

Mathematics Subject Classification: 16E50, 16U99.

Following Bass [1], a ring R has **stable range 1** if $ra + b = 1$ in R implies that $a + tb$ is a unit for some $t \in R$, equivalently $ar + b = 1$ implies $a + bt$ is a unit for some t. In this article we extend this equivalence to an arbitrary module, and use it to define what we mean by a stable module. Then, after giving a natural definition of unit regular elements in a module, we show that a module with the finite exchange property is regular stable if and only if every regular element is unit regular. This extends the result of Camillo and Yu [3, Theorem 3] that an exchange ring has stable range 1 if and only if every regular element is unit regular. Moreover, it means that the module ${}_RR$ is regular stable [5] if and only if R has internal cancellation (IC), and our approach reveals new information about these IC rings.

The proof that the stable range condition is right-left symmetric is due to Vaserstein [14]. We begin with a far-reaching generalization of Vaserstein's lemma to an arbitrary Morita context. This engenders the notion of a stable Morita context. These objects are studied in Section 1 and extend many properties of rings with stable range 1. In Section 2 all this applied to the "standard context" of a module to define and investigate stable modules. These are a natural generalization of the stable range condition for a ring, and their study leads to a generalization of rings with internal cancellation.

The notion of a unit regular element in any module is defined in Section 3, and used to give a far-reaching extension of the theorem of Camillo and Yu mentioned above: If a module ${}_RM$ has the finite exchange property then M is stable if and only if every regular element of M is unit regular. The paper concludes in Section 5 with several other properties of stable modules.

Throughout this article rings are associative with unity, modules are left modules unless otherwise specified, and morphisms will be written on the right of their arguments. The categories of left and right R-modules are denoted $R\text{mod}$ and $\text{mod}R$, respectively. We write $\text{end}(M)$ for the ring of all endomorphisms of a module M. If A and B are modules, the notation $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B. We always use $M_n(R)$ to stand for the ring of all $n \times n$ matrices over a ring R, we write $U = U(R)$ for the group of units of R, and $J = J(R)$ denotes the Jacobson radical of R. The notation $A \lhd R$ signifies that A is a (two-sided) ideal of the ring R. The term "regular ring" means "von Neumann regular ring". The left and right annihilators of set X in a set Y will be written as $\mathtt{l}_Y(X)$ and $\mathtt{r}_Y(X)$, respectively. Maps given by left and right multiplication by an element w will be denoted as $w\cdot$ amd $\cdot w$.

1. R-Stable Morita Contexts.

Morita contexts were introduced by Bass [2] in his Oregon notes on the Morita theorems. If R and S are rings, and ${}_RV_S$ and ${}_SW_R$ are bimodules, we say that the 4-tuple (R, V, W, S) is a **Morita context** if there exist multiplications

$$V \times W \to R, \text{ written } (v, w) \mapsto vw \qquad \text{and} \qquad W \times V \to S, \text{ written } (w, v) \mapsto wv$$

which induce bimodule morphisms $V \otimes_S W \to R$ and $W \otimes_R V \to S$ and satisfy

$$v(wv_1) = (vw)v_1 \quad \text{and} \quad w(vw_1) = (wv)w_1 \qquad \text{for all } v, v_1 \in V \text{ and } w, w_1 \in W.$$

These requirements are equivalent to asking that $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is an associative ring using "matrix" operations, called the **context ring**.[1] The images VW and WV are ideals of R and S respectively, called the **trace ideals** of the context. Morita proved in 1958 that R and S are (Morita) equivalent rings if and only if (in Bass' terminology) there exists a context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ such that $VW = R$ and $WV = S$ [13, Theorem A20] At the other extreme, if $V = 0$ and $W = 0$ we say that the context is **trivial.**

Notation: *For clarity, we will use* **generic notation** *when working in a context. That is, elements of a set which is denoted by an upper case letter will be denoted by the same lower case letter unless otherwise stated.*

There are many examples of Morita contexts, two important ones being $\begin{bmatrix} R & Re \\ eR & eRe \end{bmatrix}$ for any idempotent $e^2 = e \in R$, and $\begin{bmatrix} R & R^n \\ R_n & M_n(R) \end{bmatrix}$ where R^n and R_n are written as row and column matrices respectively. For us, the most important example is constructed as follows. Given a module ${}_RM$, write $S = \text{end}({}_RM)$, obtaining an R-S-bimodule $M = {}_RM_S$. The **dual** $M^* = \text{hom}(M, {}_RR)$ of ${}_RM$ is a left S-module via composition of maps, and M^* becomes a right R-module as follows: Given $\lambda \in M^*$ and $a \in R$ define $\lambda a : M \to R$ by $m(\lambda a) = (m\lambda)a$ for all $m \in M$. It is then a routine verification that $M^* = {}_S(M^*)_R$ is an S-R-bimodule. Moreover, we have multiplications:

[1] We frequently abuse the notation and refer to the context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$.

$M \times M^* \to R$ given by $(m, \lambda) \to m\lambda$ for all $m \in M$ and $\lambda \in M^*$

$M^* \times M \to S$ given by $(\lambda, m) \to \lambda m$ for all $m \in M$ and $\lambda \in M^*$ where $\lambda m : M \to M$ is defined by $x(\lambda m) = (x\lambda)m$ for all $x \in M$.

Then $\begin{bmatrix} R & M \\ M^* & \text{end}(_R M) \end{bmatrix}$ is a Morita context, called the **standard context** determined by the module $_R M$. Many of our calculations are performed in this context, but we carry them out in an arbitrary Morita context because of the clarity that generic notation brings.

Following Bass [1], a ring R is said to have **stable range 1** if it satisfies the following equivalent conditions:

S1. *$Ra + Rb = R$ implies that $ua + tb = 1$ where $t \in R$ and $u \in R$ is a unit.*
S2. *$ra + b = 1$ in R implies that $a + tb$ is a unit for some $t \in R$.*
S3. *$Ra + L = R$ where $L \subseteq R$ is a left ideal implies that $a + c$ is a unit for some $c \in L$.*

In 1984, I.N. Vaserstein [14] showed that these conditions are equivalent to their left-right analogues. More precisely, he showed that the following are equivalent for elements a, b and s in a ring R :

(1) $ab + s = 1$ in R implies $a + sx$ is a unit for some $x \in R$.
(2) $ab + s = 1$ in R implies $b + ys$ is a unit for some $y \in R$.

We begin by proving a far reaching generalization of this in an arbitrary Morita context. Moreover, the proof provides formulas for passing from each condition to the other.

Lemma 1. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context. The following are equivalent for $w \in W$ and $v \in V$:*
SC1. *If $wv + s = 1_S$, $s \in S$, then there exists $v_1 \in V$ such that $(v + v_1 s)W = R$.*
SC2. *If $wv + s = 1_S$, $s \in S$, then there exists $w_1 \in W$ such that $V(w + sw_1) = R$.*

Proof. SC1 $\Rightarrow$ SC2. Suppose that $wv+s = 1_S$, $s \in S$; we must find $w_1 \in W$ with $V(w+sw_1) = R$. By SC1 let $(v + v_1 s)W = R$ where $v_1 \in V$, say $(v + v_1 s)w' = 1_R$ where $w' \in W$. Observe that

$$vs = v - vwv = (1_R - vw)v \qquad \text{and} \qquad v_1 sw' = 1_R - vw'. \tag{*}$$

Define $v_0 = v + (1_R - vw)v_1 \in V$ and use equations (*) to compute:

$$\begin{aligned} v_0\big[sw'(1_R - v_1 w)\big] &= v\big[sw'(1_R - v_1 w)\big] + (1_R - vw)v_1\big[sw'(1_R - v_1 w)\big] \\ &= \big[(1_R - vw)v\big]w'(1_R - v_1 w) + (1_R - vw)(1_R - vw')(1_R - v_1 w) \\ &= (1_R - vw)\big[vw' + (1_R - vw')\big](1_R - v_1 w) \\ &= (1_R - vw)(1_R - v_1 w). \\ &= 1_R - (vw + v_1 w - vwv_1 w) \\ &= 1_R - v_0 w. \end{aligned}$$

Hence $v_0\big[w + sw'(1_R - v_1 w)\big] = 1_R$, and SC2 follows with $w_1 = w'(1_R - v_1 w) \in W$.

SC2 $\Rightarrow$ SC1. If $wv + s = 1_S$, we must find $w_1 \in W$ with $(v + v_1 s)w_1 = 1_R$. Condition SC2 gives $v_2(w + sw') = 1_R$ for some $v_2 \in V$ and $w' \in W$. Observe that

$$sw = w - wvw = w(1_R - vw) \qquad \text{and} \qquad v_2 sw' = 1_R - v_2 w. \tag{**}$$

Define $v_1 = (1_R - vw')v_2 \in V$, and compute using equations (**):

$$\begin{aligned} (v_1 s)\big[w + w'(1_R - vw)\big] &= (1_R - vw')v_2 s\big[w + w'(1_R - vw)\big] \\ &= (1_R - vw')\big[v_2(sw) + (v_2 s)w'(1_R - vw)\big] \\ &= (1_R - vw')\big[v_2 w(1_R - vw) + (1_R - v_2 w)(1_R - vw)\big] \\ &= (1_R - vw')(1_R - vw). \end{aligned}$$

With this, we are done with $w_1 = w + w'(1_R - vw) \in W$ because

$$(v + v_1 s)w_1 = vw_1 + v_1 s w_1 = [vw + vw' - vw'vw] + [(1_R - vw')(1_R - vw)] = 1_R. \qquad \square$$

Vaserstein's result is a special case of Lemma 1. To see this, recall that a ring R is called **directly finite** if $ba = 1$ in R implies that $ab = 1$, equivalently if $Ra = R$ implies $aR = R$.

Corollary 2. *Vaserstein's Lemma.* *If R is a ring the following are equivalent:*
(1) *$ra + b = 1$ in R implies that $a + tb$ is a unit for some $t \in R$.*
(2) *$ar + b = 1$ in R implies that $a + bt$ is a unit for some $t \in R$.*

Proof. Assume that (1) holds. Working in the context $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$, we show first that R is directly finite. If $Ra = R$ then $ra + 0 = 1$ for some $r \in R$, so Lemma 1 shows that $(a + 0s)R = R$, $s \in R$, as required. With this, to prove (2) let $ra + b = 1$ in R. Then condition SC2 in Lemma 1 shows that $(a + qb)R = R$ for some $q \in R$. It follows that $a + qb$ is a unit because R is directly finite, proving (2). This proves (1)⇒(2); the proof of (2)⇒(1) is analogous. $\square$

Definition. *A Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ will be called an R-**stable context** if the conditions in Lemma 1 are satisfied whenever $vw + s = 1_S$ in $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$.*

Corollary 3. *The following are equivalent for a ring R :*
(1) *$\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ is an R-stable Morita context.*
(2) *R has stable range 1.*
In this case R is directly finite.

Proof. (2)⇒(1) follows because Bass' condition S2 above is just SC1 for this context.

(1)⇒(2). Given (1), we show first that R is directly finite. If $Ra = R$ then $ra + 0 = 1$ for some $r \in R$, so condition SC2 in Lemma 1 shows that $(a + q0)R = R$, $q \in R$. Hence $aR = R$, as required. Now, to prove (2), let $ra + b = 1$ in R. Again SM2 shows that $(a + qb)R = R$ for some $q \in R$. Hence $a + qb$ is a unit because R is directly finite, proving (2). $\square$

Corollary 3 is as expected; moreover every R-stable context enjoys a "direct finiteness" property.

Proposition 4. *If $w_0 v_0 = 1_S$ in an R-stable context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ then $v_0 w_0 = 1_R$.*

Proof. Clearly $w_0 v_0 + 0 = 1_S$ where $0 \in S$ so condition SC2 in Lemma 1 shows that $V(w_0 + 0w_1) = R$ for some $w_1 \in W$, say $v_1 w_0 = 1_R$. Then $v_1 = v_1 1_S = v_1(w_0 v_0) = (v_1 w_0)v_0 = 1_R v_0 = v_0$. Hence $v_0 w_0 = 1_R$ as asserted. $\square$

In view of Corollary 3 and the fact that every ring with stable range 1 is directly finite, we ask:

Question 1. *Find conditions on $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ that $v_0 w_0 = 1_R$ implies $w_0 v_0 = 1_S$.*

Remark 1. *Here are four such conditions:*

(1) $\mathtt{r}_S(v_0) = 0$ *or* $\mathtt{l}_S(w_0) = 0$.

(2) ${}_SW$ *is faithful and* $\mathtt{r}_W(v_0) = 0$.

(3) V_R *is faithful and* $\mathtt{l}_V(w_0) = 0$.

(4) *The opposite context* $\begin{bmatrix} S & W \\ V & R \end{bmatrix}$ *is* S*-stable.*

Definition. *We say that* $v_0 \in V$ *is* **right invertible** *in a Morita context* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *if* $v_0w_0 = 1_R$ *for some* $w_0 \in W$ *(equivalently if* $v_0W = R$*); we say* v_0 *is* **left invertible** *if* $w_0v_0 = 1_S$ *for some* $w_0 \in W$ $(Wv_0 = S)$*; and we say that* $v_0 \in V$ *is* **invertible** *if it is both right and left invertible (equivalently if there exists* $w_0 \in W$ *such that both* $v_0w_0 = 1_R$ *and* $w_0v_0 = 1_S$*). Analogously, we speak of* $w_0 \in W$ *being (right, left) invertible.*

These notions are important here as SC1 and SC2 in Lemma 1 assert respectively that $v + v_1s$ is right invertible and $w + sw_1$ is left invertible.

Lemma 5. *Let* $v_0w_0 = 1_R$ *in a Morita context* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$. *Write* $f = w_0v_0$ *so that* $f^2 = f \in S$. *Then*:

(1) ${}_RV$ *is faithful,* ${}_RV = Rv_0 \oplus \mathtt{l}_V(w_0)$, $\quad Rv_0 = Vf \cong {}_RR \quad$ *and* $\quad \mathtt{l}_V(w_0) = V(1_S - f)$.

(2) W_R *is faithful,* $W_R = w_0R \oplus \mathtt{r}_W(v_0)$, $\quad w_0R = fW \cong R_R \quad$ *and* $\quad \mathtt{r}_W(v_0) = (1_S - f)W$.

(3) $V_S = v_0S \cong fS \quad$ *and* $\quad {}_SW = Sw_0 \cong Sf$.

Proof. (1). ${}_RV$ is faithful because $rV = 0$ implies $r = r1_R = rv_0w_0 = 0$

$W_R = w_0R \oplus \mathtt{r}_W(v_0)$: First, $V = Rv_0 + \mathtt{l}_V(w_0)$ because $v_1 - v_1w_0v_0 \in \mathtt{l}_V(w_0)$ for any $v_1 \in V$. If $rv_0 \in \mathtt{l}_V(w_0)$ then $0 = (rv_0)w_0 = r(v_0w_0) = r$, so $rv_0 = 0$. This shows that $Rv_0 \cap \mathtt{l}_V(w_0) = 0$.

Next, $Rv_0 = Vf$ because $Rv_0 = R(v_0w_0)v_0 \subseteq Vf$, and $Vf = V(w_0v_0) = (Vw_0)v_0 \subseteq Rv_0$.

We have $Rv \cong {}_RR$ because $R \overset{\cdot v_0}{\to} Rv_0$ is an R-isomorphism.

Finally $\mathtt{l}_V(w_0) \subseteq V(1_S - f)$ because $vw_0 = 0$ implies that $vf = vw_0v_0 = 0$ so $v = v(1_S - f)$. Conversely, $V(1_S - f) \subseteq \mathtt{l}_V(w_0)$ because $(1_S - f)w_0 = w_0 - w_0v_0w_0 = 0$.

(2). The proof is analogous to that of (1).

(3). We have $V = 1_RV = v_0w_0V \subseteq v_0S$, so $V = v_0S$. And the map $v_0W \to fS$ given by $v_0s \mapsto fs$ is well defined because $v_0s = 0$ implies $fs = w_0v_0s = 0$, and so is S-linear. It is monic because $fs = 0$ implies $v_0s = (v_0w_0)v_0s = v_0(fs) = 0$. This proves $V_S = v_0S \cong fS$. The proof that ${}_SW = Sw_0 \cong Sf$ is analogous. □

Theorem 6. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be any* R*-stable Morita context. Then there exist* $v_0 \in V$ *and* $w_0 \in W$ *such that* $v_0w_0 = 1_R$.

Proof. Write $v' = 0$ in V and $w' = 0$ in W. Then $w'v' + 1_S = 1_S$ so, by SC1, there exists $v_1 \in V$ such that $(v' + v_11_S)W = R$, that is $v_1W = R$, say $v_1w_1 = 1_R$. Now (1) follows with $v_0 = v_1$ and $w_0 = w_1$. □

Recall that a module ${}_RG$ is called a **generator** for the category R mod of all left R-modules if every left R-module is an image of a direct sum of copies of ${}_RG$, equivalently if ${}_RR$ is isomorphic to a direct summand of G^n for some $n \geq 1$. Combining Theorem 6 and Lemma 5 gives:

Corollary 7. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be R-stable. Then:*

(1) ${}_RV$ *and* W_R *are faithful generators in* $R\,\mathrm{mod}$ *and* $\mathrm{mod}\,R$, *respectively.*

(2) V_S *and* ${}_SW$ *are principal, projective S-modules.*

Proof. (1). By Theorem 6 and Lemma 5 we have $v_0 \in V$ and $w_0 \in W$ such that ${}_RV = Rv_0 \oplus \mathtt{l}_V(w_0)$ and $Rv_0 \cong {}_RR$. This shows that ${}_RV$ is a generator in $R\,\mathrm{mod}$. Similarly, $W_R = w_0R \oplus \mathtt{r}_W(v_0)$ and $w_0R \cong R_R$ show that W_R is a generator in $\mathrm{mod}\,R$.

(2). These follow from (3) of Lemma 5. □

Theorem 6 and Lemma 5 also give:

Corollary 8. *Let* $v_0w_0 = 1_R$ *in the R-stable context* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, *and write* $f = w_0v_0 \in S$. *Then:* $f^2 = f$, $VW = R$, $WV = SfS$, $w_0Rv_0 = fSf$ *and* $v_0Sw_0 = R$.

Proof. We use Lemma 5 repeatedly and without comment.

(1). $VW = R$ because $1_R \in VW$, and $WV = (Sf)(fS) = SfS$. To see that $w_0Rv_0 = fSf$, observe that: $w_0Rv = w_0(v_0w_0)R(v_0w_0)v_0 = f(w_0Rv_0)f \subseteq fSf$, and $fSf = w_0(v_0Sw_0)v_0 \subseteq w_0Rv_0$. Finally, $R \subseteq v_0Sw_0$ because $r = (v_0w_0)\,r(v_0w_0) \in v_0Sw_0$ for all $r \in R$; the other inclusion is clear. □

Corollary 9. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be R-stable. Suppose* $\mathfrak{p}$ *is a property of modules that is inherited by direct summands.*

(1) *If* ${}_RV$ *has* $\mathfrak{p}$ *then* ${}_RR$ *has* $\mathfrak{p}$.

(2) *If* W_R *has* $\mathfrak{p}$ *then* R_R *has* $\mathfrak{p}$

Proof. Suppose ${}_RV$ has $\mathfrak{p}$. By Theorem 6 and Lemma 5 ${}_RR$ is isomorphic to a direct summand of ${}_RV$, and so has $\mathfrak{p}$ by hypothesis. This proves (1), and (2) is similar. □

A module ${}_RM$ is called **semiperfect** if it is projective and every image has a projective cover. And M is **semiregular** if M/K has a projective cover for every finitely generated (respectively principal) submodule K—see [12] 0r [13, Page 266 and 280]. We now list some examples using Corollary 9.

Example 10. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be an R-stable Morita context.*

(1) *If* ${}_RV$ *is semisimple then* R *is a semisimple ring.*

(2) *If* ${}_RV$ *is simple then* R *is a division ring.*

(3) *If* ${}_RV$ *is semiperfect then* R *is a semiperfect ring.*

(4) *If* ${}_RV$ *is semiregular then* R *is a semiregular ring.*

(5) *If* ${}_RV$ *is self-injective then* R *is a left self-injective ring.*

(6) *If* ${}_RV$ *is artinian (respectively noetherian) then* ${}_RR$ *has the same property.*

Moreover, the analogues for W_R *are also true.*

Proof. (1). Taking semisimplicity passes to direct summands.

(2). If ${}_RV$ is simple then ${}_RR$ is simple by in Lemma 5.

(3). Every direct summand of a semiperfect module is again semiperfect (being an image), and R is a semiperfect ring if and only if ${}_RR$ is semiperfect.

(4). As in (3), this follows from the fact that direct summands of semiregular modules are semiregular by [12, Theorem 1.10], and R is a semiregular ring if and only if ${}_RR$ is semiregular.

(5). Injectivity passes to direct summands.

(6). These properties are inherited by summands, and ${}_RR$ is left artinan (left noetherian) if and only if R has the same property as a ring. □

Verifying that $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is R-stable is much easier if S is an exchange ring, because in that case condition SC1 need only be verified for *idempotents* $s \in S$.

Theorem 11. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be a Morita context where* S *is an exchange ring. The following are equivalent:*

(1) $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is* R*-stable.*

(2) *If* $wv + f = 1_S$ *where* $f^2 = f \in S$*, then there exists* $v_1 \in V$ *such that* $(v + v_1 f)W = R$*.*

Proof. (1)⇒(2) is clear.

(2)⇒(1). If $wv + s = 1_S$, $s \in S$; we must show that $(v + v_1 s)$ is right invertible for some $v_1 \in V$. Since S is exchange, there exists $f^2 = f \in Ss$ such that $1_S - f \in S(1_S - s)$ [11]. Hence $1_S - f \in Swv \subseteq Wv$, say $1_S - f = w_1 v$, $w_1 \in W$. Thus $w_1 v + f = 1_S$ so, by (2), let $(v + v_2 f)W = R$, $v_2 \in V$. But $f \in Ss$, say $f = s_2 s$, so $v + (v_2 s_2)s = v + v_2 f$ is right invertible, as desired. □

We note in passing that Theorem 11 holds if the set of idempotents in S is replaced by any subset $X \subseteq S$ with the property that if $wv + x = 1_S$ where $x \in X$, then $(v + v_1 x)W = R$ for some $v_1 \in V$.

Applying Theorem 11 to the context $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ yields an important result of Camillo and Yu:

Corollary 12. *[3, Lemma 2] An exchange ring* R *has stable range 1 if and only if* $ra + e = 1$*,* $r, a, e = e^2$ *in* R *implies that* $a + te$ *is a unit for some* $t \in R$*.*

Proof. R is directly finite by the proof of Corollary 3, and the result follows by theorem 11. □

Question 2. *Call a ring* **idempotent stable** *if* $ra + e = 1$*,* $r, a, e = e^2$ *in* R *implies that* $a + te$ *is a unit for some* $t \in R$*. When is an idempotent stable ring an exchange ring?*

2. Stable and Regular Stable Modules

Our main interest here is in studying stable modules ${}_RM$ as a generalization of rings with stable range 1. Here is the definition.

Definition. *We say a module* ${}_RM$ *is a* **stable module** *if the standard context* $\begin{bmatrix} R & M \\ M^* & \mathrm{end}(M) \end{bmatrix}$ *is* R*-stable, that is if the following equivalent statements are true for all* $m \in M$ and $\lambda \in M^*$:

SM1 *If* $\lambda m + \theta = 1_M$ *where* $\theta \in \mathrm{end}(M)$*, there exists* $q \in M$ *such that* $(m + q\theta)M^* = R$*.*

SM2 *If* $\lambda m + \theta = 1_M$ *where* $\theta \in \mathrm{end}(M)$, *there exists* $\mu \in M^*$ *such that* $M(\lambda + \theta\mu) = R$.

In the spirit of generic notation, we usually denote elements in a module M, the dual M^*, and the endomorphism ring $\mathrm{end}(M)$ by (variants of) m, λ and θ, respectively.

In the important case of the module ${}_RR$ we identify $({}_RR)^* = R = \mathrm{end}({}_RR)$ so the standard context for ${}_RR$ is $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$. Hence Corollary 3 gives:

Corollary 13. *The following are equivalent for a ring* R :
(1) ${}_RR$ *is a stable module.*
(2) R *has stable range* 1.
(3) R_R *is a stable module.*

The context ring $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ has stable range 1 if and only if both R and S have stable range 1 $\Big[$if $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then $R \cong eRe$ and $S \cong (1-e)S(1-e)\Big]$.

Question 3. *If* ${}_RM$ *is stable, when does the context ring* $\begin{bmatrix} R & M \\ M^* & S \end{bmatrix}$ *have stable range* 1?

Example 14. *Let* ${}_{\mathbb{Z}}V = \mathbb{Z} \oplus \mathbb{Z}$ *and write* $S = \mathrm{end}({}_{\mathbb{Z}}V)$. *Then* ${}_{\mathbb{Z}}V$ *is a stable module but* ${}_{\mathbb{Z}}\mathbb{Z}$ *is not, so direct summands of stable modules need not be stable. Thus the standard context* $\begin{bmatrix} \mathbb{Z} & V \\ V^* & S \end{bmatrix}$ *is* $\mathbb{Z}$*-stable but* $\mathbb{Z}$ *does not have stable range 1.*

Proof. Write ${}_{\mathbb{Z}}M = \mathbb{Z} \oplus \mathbb{Z}$. Given $\lambda m + \theta = 1_M$ where $\lambda \in M^*$, $m \in M$ and $\theta \in \mathrm{end}(M)$, we define $\delta : \mathbb{Z} \to M$ by $x\delta = xm$. Then $\lambda\delta + \theta = 1_M$. As $\mathrm{end}({}_{\mathbb{Z}}\mathbb{Z})$ has stable range 2, it follows from [4, Lemma 12.1.1] that there exist $\alpha \in M^*$ and $\beta \in \hom_{\mathbb{Z}}(\mathbb{Z}, M)$ such that $\beta(\lambda + \theta\alpha) = 1_{\mathbb{Z}}$. This shows that $(1\beta)(\lambda + \theta\alpha) = 1$, so $M(\lambda + \theta\alpha) = \mathbb{Z}$. Hence M is stable. □

Question 4. *Is the direct sum of two stable modules again stable*?

Example 15. *If* M *is a nil left ideal of* R, *then* ${}_RM$ *is not a stable module. However the standard context* $\begin{bmatrix} R & J \\ J^* & \mathrm{end}({}_RJ) \end{bmatrix}$ *may be* R*-stable where* $J = J(R)$.

Proof. Suppose ${}_RM$ is stable, and let $m \in M$ and $\lambda = 0$ in M^*. Then $\lambda m + 1_M = 1_M$ so SM1 gives $R = M(\lambda + 1_M\mu) = M\mu$ for some $\mu \in M^*$. Let $u\mu = 1$, $u \in M$. As M is nil write $u^k = 0$, $k \geq 1$, where $u^{k-1} \neq 0$. Then $u^{k-1} = u^{k-1}(u\mu) = u^k\mu = 0$, a contradiction.

On the other hand, let $R = \mathbb{Z}_{(2)} = \{\frac{k}{d} \in \mathbb{Q} \mid k, d \in \mathbb{Z} \text{ and } d \text{ is odd}\}$ denote the localization of $\mathbb{Z}$ at 2. Then $\begin{bmatrix} R & M \\ M^* & \mathrm{end}(M) \end{bmatrix}$ is an R-stable context by Corollary 13 because R has stable range 1 (R is local). Here ${}_RJ = 2R \cong {}_RR$ via $r \mapsto 2r$ so ${}_RJ$ is stable. □

If we specialize Theorem 6 and its corollaries to the standard context of a module, we obtain:

Theorem 16. *Let* ${}_RM$ *be a stable module and write* $S = \mathrm{end}(M)$. *Then*:
(1) *There exist* $\lambda_0 \in M^*$ *and* $m_0 \in M$ *such that* $\lambda_0 m_0 = 1_M$.
Write $f = m_0\lambda_0 \in S$ *so that* $f^2 = f$. *Then*:

(2) *$MM^* = R$ and $M^*M = SfS$.*
(3) *${}_RM$ and $(M^*)_R$ are faithful generators of $R\,\mathrm{mod}$ and $\mathrm{mod}\,R$, respectively.*
(4) *M_S and ${}_SM^*$ are principal, projective modules.*
(5) *${}_RM = Rm_0 \oplus \mathtt{l}_M(\lambda_0)$ and $(M^*)_R = \lambda_0 R \oplus r_{M^*}(m_0)$.*
(6) *$M_S = m_0 S \cong \theta_0 S$ and ${}_SM^* = S\lambda_0 \cong S\theta_0$.*
(7) *Given a property of modules that is inherited by direct summands, then if ${}_RM$ (respectively $(M^*)_R$) has the property the same is true of ${}_RR$ (respectively R_R).*

Thus, for example, if ${}_RM$ is stable and artinian, noetherian, injective, semisimple or semiregular then R has the same property as a ring.

While stable modules are interesting, a less restrictive condition is better when capturing many of the properties of stable rings. Recall that an element a in a ring R is called (*unit*) *regular* if $a = aba$ for some element (some unit) b in R. In 1972 Zelmanowitz [16] called an element m in a module ${}_RM$ **regular** if $(m\lambda)m = m$ for some $\lambda \in M^*$, and he called M a **regular module** if every element is regular. We extend these notions to an arbitrary Morita context as follows:

Definition. *If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is a Morita context, an element $v \in V$ is called* **regular** *if $vwv = v$ for some $w \in W$. Similarly, $w \in W$ is called regular if $wvw = w$ for some $v \in V$.*

Lemma 1 shows that the conditions SC1 and SC2 are equivalent for all $v \in V$, $w \in W$ and $s \in S$ in an R-stable Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$. We weaken this requirement by insisting only that it hold for all *regular* v (and all w and s). More precisely:

Definition. *A Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is called* **regular** R**-stable** *if the following equivalent conditions hold for $v \in V$, $w \in W$ and $s \in S$:*

RSC1. *If $wv + s = 1_S$ where v is regular then $(v + v_1 s)W = R$ for some $v_1 \in V$.*

RSC2. *If $wv + s = 1_S$ where v is regular then $V(w + sw_1) = R$ for some $w_1 \in W$.*

Because $v = 0$ is a regular element of V, the proof of Theorem 6 goes through for any regular R-stable context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$. Consequently, Corollaries 7, 8 and 9 all hold for any regular R-stable context. They are collected in the next result for reference.

Theorem 17. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be regular R-stable. Then:*
(1) *There exist $v_0 \in V$ and $w_0 \in W$ such that $v_0 w_0 = 1_R$.*
(2) *${}_RV$ and W_R are faithful generators in $R\,\mathrm{mod}$ and $\mathrm{mod}\,R$, respectively.*
(3) *V_S and ${}_SW$ are principal, projective S-modules.*
Now write $f = w_0 v_0 \in S$. Then:
(4) *$f^2 = f$, $VW = R$, $WV = SfS$, $w_0 R v_0 = fSf$ and $v_0 S w_0 = R$.*
(5) *If $\mathfrak{p}$ is a property of modules that is inherited by direct summands, then*
(*a*) *If ${}_RV$ has $\mathfrak{p}$ then ${}_RR$ has $\mathfrak{p}$.*
(*b*) *If W_R has $\mathfrak{p}$ then R_R has $\mathfrak{p}$.*

A module M is said to have **internal cancellation** (**IC**) if, whenever M has two submodule decompositions $M = K \oplus N = L \oplus N'$ such that $N \cong N'$, then $K \cong L$. In 1976 Ehrlich [5] proved the following theorem: Given a module $_RM$ with $\mathrm{end}(M)$ regular, then M has IC if and only if $\mathrm{end}(M)$ is unit regular. In fact all she needed was that every regular element of $\mathrm{end}(M)$ is unit regular. In [9] Khurana and Lam called this latter result Ehrlich's theorem, and they coined the term **IC-ring** for any ring in which every regular element is unit regular. With an eye on this, we consider the modules where SM1 and SM2 hold for all *regular* elements m. More precisely:

Definition. *Let $_RM$ be a module and write $S = \mathrm{end}(M)$. Then M is called* **regular-stable** *if the standard context $\begin{bmatrix} R & M \\ M^* & S \end{bmatrix}$ is a regular R-stable. In other words, given $\lambda \in M^*$ and $\theta \in E$ the following equivalent statements hold*:

RSM1 *If $m \in M$ is regular and $\lambda m + \theta = 1_M$ then $M(\lambda + \theta\lambda_1) = R$ for some $\lambda_1 \in M^*$.*

RSM2 *If $m \in M$ is regular and $\lambda m + \theta = 1_M$ then $(m + m_1\theta)M^* = R$ for some $m_1 \in M$.*

We are going to describe the rings R where $_RR$ is regular R-stable. To this end, call an element $a \in R$ **stable** if $ra + b = 1$ in R implies that $a + tb \in U = U(R)$ for some $t \in R$.

Lemma 18. *If R is any ring, the product of two stable elements is again stable. Hence the set of stable elements is a submonoid of R that contains all units and idempotents, and which equals R if and only if R has stable range* 1.

Proof. Let a and a' be stable, and assume that $ra'a + b = 1$ in R. Since a is stable, it follows that $a + tb \doteq u \in U(R)$ for some $t \in R$. Hence $1 = ra'(u - tb) + b = ra'u + xb$, $x \in R$. Conjugating by u gives $1 = ura' + uxbu^{-1}$. As a' is stable we obtain $a' + t'bu^{-1} \doteq u_1 \in U(R)$ where $t' \in R$. Hence $a'u + t'b = u_1u$ so, since $u = a + tb$, $a'a + (at + t')b = u_1u \in U(R)$, proving that aa' is stable. □

Proposition 19. *If R is a ring, let $\mathrm{s}(R)$ denote the set of all stable elements in R. Then*:

(1) *$\mathrm{s}(R)$ is a multiplicative submonoid of R containing every unit regular element of R.*

(2) *$\mathrm{s}(R) = R$ if and only if R has stable range 1.*

Proof. (1). Units $u \in U$ are left stable $\big[$if $ru + b = 1$ then $u + 0b \in U\big]$; in particular $\mathrm{s}(R)$ is a multiplicative submonoid of R by Lemma 18. Also idempotents e are left stable $\big[$if $re + b = 1$ then $e + (1 - e)b = 1 - (1 - e)re \in U\big]$. Hence unit regular elements a are left stable $\big[$if $aua = a$ where $u \in U$ then $a = eu^{-1}$ where $e^2 = e = au\big]$. This proves (1); (2) is clear. □

In particular, this makes it clear that every unit regular ring has stable range 1, an unpublished result of Kaplansky (see [6]).

Theorem 20. *The following are equivalent for a ring R* :

(1) *$_RR$ is a regular stable module.*

(2) *If $ra + b = 1$ in R and a is regular, then $a + tb$ is a unit for some $t \in R$.*

(3) *If $ar + b = 1$ in R and a is regular, then $a + bt$ is a unit for some $t \in R$.*

(4) *R_R is a regular stable module.*

(5) *R is an IC ring, that is every regular element in R is unit regular.*

In this case R is directly finite.

Proof. As before, we identify $({}_RR)^* = R = \mathrm{end}({}_RR)$.

(1)⇔(2). Condition (2) implies condition SM2 for the module ${}_RR$, so (2)⇒(1). Conversely, assume (1) holds. As before, we show first that R is directly finite. Suppose $Ra = R$, so that $ra + 0 = 1$ for some $r \in R$. But a is regular because $Ra = R$, so (1)–and hence SM2–shows that $(a + q0)R = R$ for some $\mu \in R$. Hence $aR = R$ as required. Now, to prove (2), let $ra + b = 1$ in R where a is regular. Then SM2 shows that $(a + qb)R = R$ for some $q \in R$. It follows that $a + qb$ is a unit because R is directly finite, proving (2).

(2)⇔(3). This is by Lemma 1.

(3)⇔(4). This is analogous to (1) ⇔ (2).

(2)⇒(5). Let $axa = a$. Then $xa + (1 - xa) = 1$ so (2) shows $a + t(1 - xa) = u \in U(R)$ for some $t \in R$. Hence $u(xa) = a(xa) + t0 = a$, so $xa = u^{-1}a$. But then $au^{-1}a = a$, proving (5).

(5)⇒(2). Let $ra + b = 1$ where a is regular. Then a is unit regular by (5), and so is stable by Lemma 18. □

Example 21. *The ring $\mathbb{Z}$ of integers is an IC ring since the only unit regular elements are* 0, 1 *and* -1. *However $\mathbb{Z}$ does not have stable range 1 by Example 14.*

In [9] Khurana and Lam gave a thorough study of IC rings and presented other characterizations of these rings. In particular, they point out that the class of IC rings is large, including all abelian rings (central idempotents), all unit-regular rings, all one-sided artinian rings, all CS rings and (from operator theory) all finite von Neumann algebras. Moreover, if R is IC so also is any corner eRe [9, Proposition 5.6]. However, the IC property does not pass to matrix rings [9, Example 5.9(2) and 5.9(3)]; the earliest example we know is given in [7, Page 50].[2]

3. Unit Regular Elements in a Morita Context

A theorem of Camillo and Yu [3] asserts that an exchange ring R is stable if and only if every regular element of R is unit regular (that is, R is an IC ring). We are going to prove an extension of this results to modules with the finite exchange property, but we must first define the "unit regular" elements in a module. To do this we need the following basic lemma presenting one situation where condition SC2 is always satisfied.

Lemma 22. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be a Morita context. Suppose $v \in V$ has the form $v = v_1 f$ where v_1 is right invertible and $f^2 = f \in S$. Then SC2 holds for v and any $w \in W$, that is:*

If $wv + s = 1_S$, $w \in W$, $s \in S$, then $(v + v_2 s)W = R$ for some $v_2 \in V$.

Proof. Define $s_1 = f + (1_S - f)s \in S$. Since $v = v_1 f$ we obtain

$$s_1 = f + (1_S - f)(1_S - wv) = 1_S - (1_S - f)wv = 1_S - (1_S - f)wv_1 f.$$

Hence $s_1 \in \mathrm{aut}({}_RV)$ because $f^2 = f$. Now write $v_2 = v_1(1_S - f)$, and observe that

$$v + v_2\theta = v_1 f + v_1(1_S - f)s = v_1[f + (1_S - f)s] = v_1 s_1.$$

Since v_1 is right invertible, $(v + v_2 s)W = v_1 s_1 W = v_1 W = R$, proving Lemma 22. □

With this we can identify the "unit regular" elements in any Morita context.

[2]Note that IC rings are denoted as p.u.r. rings in [7].

Lemma 23. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be any Morita context. The following are equivalent for* $v \in V$:

(1) $v = (vw)v$ *for some left invertible* $w \in W$.

(2) v *is regular and* $(wv)^2 = wv \in S$ *for some left invertible* $w \in W$.

(3) v *is regular and* $v = v_1 f$ *for some right invertible* $v_1 \in V$ *and* $f^2 = f \in S$.

Proof. (1)⇒(2). If w is as in (1), then $(wv)^2 = w(vwv) = wv$ in S.

(2)⇒(3). Choose w as in (2), and write $f = wv$. Then $f^2 = f \in S$. Since $w \in W$ is left invertible, let $v_1 w = 1$, $v_1 \in V$. Then v_1 is right invertible, and $v_1 f = v_1(wv) = (v_1 w)v = 1v = v$.

(3)⇒(1). As in (3), let $v = v_1 f$ where $v_1 W = R$ and $f^2 = f \in E$. We utilize Lemma 1. Since v is regular by (3), let $(vw)v = v$ where $w \in W$. Write $s = 1_S - wv$, so that $wv + s = 1_S$. Hence Lemma 22 shows that condition SC2 is satisfied for this w and s. But then (by Lemma 1) SC1 is also satisfied, that is $V(w + sw_2) = R$ for some $w_2 \in W$. Writing $w_1 = w + sw_2 \in W$ this means that w_1 is left invertible. But $vs = v - vwv = 0$, so $(vw_1)v = (vw)v = v$, proving (1). □

Definition. *If* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is a Morita context, call* $v \in V$ **unit regular** *if the conditions in Lemma 23 are satisfied.*

Combining (3) of Lemma 23 with Lemmas 22 and 1 gives:

Corollary 24. *If* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *is a Morita context and* $v \in V$ *is unit regular then SC1 and SC2 hold for* v *and any* $w \in W$.

Note that there exist regular elements a in a ring R that satisfy $a = aua$ where u has a one-sided inverse, but are not unit regular. For example, let $R = \text{end}(V)$ where ${}_F V$ is a vector space over a field F with basis $\{v_0, v_1, v_2, \ldots\}$, and let $\alpha \in R$ be the shift operator defined by $v_i \alpha = v_{i+1}$ for each i. If $\beta \in R$ is defined by $v_0 \beta = 0$ and $v_i \beta = v_{i-1}$ for all $i \geq 1$, then $\alpha\beta = 1_V$. Hence $\alpha\beta\alpha = \alpha$, $\beta\alpha\beta = \beta$, β has a left inverse and α has a right inverse. But neither α nor β is unit regular by a result of Ehrlich [5, Theorem 1]: An endomorphism $\gamma : M \to M$ is unit regular if and only if $\ker(\gamma)$ and $M\gamma$ are both direct summands of M and $M/M\gamma \cong \ker(\gamma)$.

We can now characterize the regular R-stable modules in a way that extends the result that ${}_R R$ is regular R-stable if and only if R is an IC ring (in Theorem **??**).

Theorem 25. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be any Morita context. The following are equivalent*:

(1) *The context is regular* R*-stable.*

(2) *If* $v \in V$ *is regular then* v *is unit-regular.*

Proof. (1)⇒(2). Let $v \in V$ be regular, say $v = (vw)v$, $w \in W$, and write $s = 1_S - wv \in S$. By (1), SC1 holds for this w and v so, since $wv + s = 1_S$, we have $V(w + sw_1) = R$ for some $w_1 \in W$. If we write $w_2 = w + sw_1$, this means that w_2 is left invertible. But $vw_2 = vw$ because $vs = 0$, so $(vw_2)v = v$, proving (2).

(2)⇒(1). If $v \in V$ is regular and $w \in W$, it suffices to show that SC2 holds for this v and w. Since w is unit regular by (2), use Lemma 23 to write $v = v_1 \pi$ where $v_1 \in V$ is right invertible and ${s_0}^2 = s_0 \in E$. Suppose that $wv + s = 1_S$ where $s \in S$. If we write $s_1 = s_0 + (1_S - s_0)s$ we have

$$s_1 = s_0 + (1_S - s_0)(1_S - wv) = 1_S - (1_S - s_0)wv = 1_S - (1_S - s_0)wv_1s_0.$$

Hence $\sigma \in \text{aut}({}_RV)$ because ${s_0}^2 = \pi$. But then $v + v_1(1_M - s_0)s = v_1s_1 \in V$ is also right invertible because $v_1s_1W = v_1W = R$. Hence SC2 is satisfied with $v_2q = v_1(1_S - s_0)$, proving (1). ☐

Applying Theorem 25 to the standard context of a module ${}_RM$ gives:

Theorem 26. *Let ${}_RM$ be a module with $E = \text{end}(M)$. Then the following are equivalent*:
(1) *M is R-stable.*
(2) *If $m \in M$ is regular then m is unit-regular.*

Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context and consider $V^* = ({}_RV)^* = \text{hom}({}_RV, {}_RR)$, the dual of the module ${}_RV$. Then V^* becomes an S-R-bimodule as follows: If $\lambda \in V^*$, $r \in R$ and $s \in S$ define

$$\lambda r : V \to R \text{ by } v(\lambda r) = (v\lambda)r \text{ for all } v \in V, \quad \text{and}$$
$$s\lambda : V \to R \text{ by } v(s\lambda) = (vs)\lambda \text{ for all } v \in V.$$

This bimodule is related to ${}_SW_R$. Indeed, the map $\theta : W \to V^*$ defined by $\theta(w) = \cdot w$ for all $w \in W$ is an R-S-bimodule morphism with $\ker(\theta) = \{w \in W \mid Vw = 0\}$.

An element $a \in R$ is unit regular if and only if a is the product of a unit and an idempotent (in either order). The next result gives a useful form of this in any context, which is equivalent to unit regularity for the standard context of a module.

Theorem 27. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context in which the following condition holds*:

Each R-epimorphism $\lambda : {}_RV \to R$ has the form $\lambda = \cdot w$ for some $w \in W$. (†)

Given $v \in V$ the following are equivalent:[3]
(1) *v is unit regular.*
(2) *$v = (vw)v = (vw)v_0 = v_0(wv)$ where $w \in W$ and $v_0 \in V$ is right invertible.*

Proof. (1)⇒(2). Let $v \in M$ be unit regular, say $(vw_1)v = v$ where $w_1 \in W$ is left invertible. Define $w = (w_1v)w_1$ so that $vw_1v = v$ and $w_1vw_1 = w_1$. Write $s = 1_S - wv \in S$. Then $wv + s = 1_S$ so, since v is unit regular, Corollary 24 implies that $v_1 = v + v_2s \in V$ is right invertible for some $v_2 \in V$. We have

$$v_1w = vw \text{ (because } sw = 0) \quad \text{so} \quad wv_1w = wvw = w.$$

Write $r = 1_R - 2v_1w \in R$ and $s_1 = 1_S - wv - wv_1 \in S$. Then $r^2 = 1_R$ because $(v_1w)^2 = v_1w$, and also

$${s_1}^2 = 1_S - 2(wv + wv_1) + (wvwv + wvwv_1 + wv_1wv + wv_1wv_1) = 1_S.$$

Write $v_0 = rv_1s_1 \in V$. Then v_0 is also right invertible because $v_0W = rv_1(s_1W) = rv_1W = rR = R$. Moreover, since $wv_1w = wvw = w$ we have $s_1w = -w = wr$, so we obtain

$$v_0w = (rv)s_1w = rv_1(-w) = -r(v_1w) = -[v_1w - 2(v_1w)^2] = v_1w = vw, \quad \text{and}$$
$$wv_0 = w(rv_1s_1) = (-w)v_1s_1 = -(wv_1)s_1 = -wv_1 + wv_1wv + wv_1wv = wv.$$

These give, respectively, that $v_0wv = (vw)v = v$ and $vwv_0 = v(wv) = v$, proving (2).

[3]Note that (1)⇒(2) does not require (†).

(2)⇒(1). As in (2), let $v = (vw)v = (vw)v_0 = v_0(wv)$ where $w \in W$ and $v_0 \in V$ is right invertible. Let $w_1 \in W$ be such that $v_0 w_1 = 1_R$. Since wv and vw are idempotents, we have

$$V = Vwv \oplus V(1_S - wv) \quad \text{and} \quad R = Rvw \oplus R(1_R - vw).$$

Using these decompositions, construct two maps $\gamma : V \to R$ and $\phi : R \to V$ as follows:

$$\big[pwv + q(1_S - wv)\big]\gamma = (pwv)w + \big[q(1_S - wv)\big]w_1(1_R - vw) \text{ for any } p, q \in V.$$
$$\big[rvw + s(1_R - vw)\big]\phi = (rvw)v + \big[s(1_R - vw)\big]v_0(1_S - wv) \text{ for any } r, s \in R.$$

CLAIM. $\phi\gamma = 1_R$.

Proof. We examine the components separately. The first is easy:

$$rvw \overset{\phi}{\mapsto} (rvw)v = (rv)wv \overset{\gamma}{\mapsto} (rv)wvw = rvw.$$

Next observe: $v_0(1_S - wv)w_1 = 1_R - (v_0 wv)w_1 = 1_R - (vwv_0)w_1 = 1_R - vw$. With this the second component reads

$$\begin{array}{rcccl} s(1_R - vw) & \overset{\phi}{\mapsto} & \big[s(1_R - vw)\big]v_0(1_S - vw) & \overset{\gamma}{\to} & s(1_R - vw)v_0(1_M - vw)w_1(1_R - wv) \\ & & & = & s(1_R - vw)^3 \\ & & & = & s(1_R - vw). \end{array}$$

This proves the Claim.

Now observe that $v\gamma = (vwv)\gamma = (vwv)w = vw$, so $(v\gamma)v = v$. Since $\gamma : {}_RV \to R$ is epic by the Claim, (†) asserts that $\gamma = \cdot w_0$ for some $w_0 \in W$, whence $vw_0v = v$. So it remains to show that w_0 is left invertible. But if we write $v_1 = 1_R\phi$, then $v_1 w_0 = (1_R\phi)\gamma = 1_R$ by the Claim, as required.□

Question 5. *Is* (†) *in Theorem* 28 *satisfied in every R-stable Morita context* $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix}\right]$?

Since condition (†) is clearly satisfied in the standard context of a module ${}_RM$, we obtain:

Theorem 28. *The following are equivalent for an element m in a module ${}_RM$.*

(1) *m is unit regular.*

(2) *$m = (m\lambda)m = (m\lambda)u = u(\lambda m)$ for some $\lambda \in M^*$ and right invertible $u \in M$.*

4. Analogue of the Camillo-Yu Theorem

A theorem of Kaplansky (unpublished, see [3, Theorem 3]) states that a regular ring has stable range 1 if and only if it is unit-regular. In 1995 Camillo and Yu proved an important generalization for exchange rings. More precisely, they showed that an exchange ring R has stable range 1 if and only if every regular element of R is unit regular. To generalize this to modules, recall that a module M has the **finite exchange property** if, for every module X and two decompositions

$$X = M' \oplus N = \textstyle\bigoplus_{i \in I} N_i, \text{ where } M' \cong M \text{ and the index set } I \text{ is finite,}$$

there exist submodules $N_i' \subseteq N_i$ for each i such that $N = M' \oplus \big(\bigoplus_{i \in I} N_i'\big)$. If R is a ring, Warfield [15] called R an **exchange ring** if ${}_RR$ has the finite exchange property, and showed that this property is left-right symmetric. He also proved that a module M has the finite exchange property if and only if its endomorphism ring is an exchange ring.

The Camillo-Yu theorem asserts that if R is an exchange ring then R has stable range 1 if and only if every regular element of R is unit regular, that is if and only if R is an IC ring. Consequently, (1)⇔(2) in the next result is a far-reaching extension.

Theorem 29. *Given ${}_RM$, write $S = \text{end}(M)$. If ${}_RM$ has the finite exchange property, the following are equivalent*:

(1) *M is stable.*

(2) *Every regular element of M is unit regular.*

(3) *M is regular R-stable.*

(4) *If $\lambda m + \theta = 1_S$, $\lambda \in M^*$, $m \in M$, $\theta^2 = \theta \in S$, then $M(\lambda + \pi\lambda_1) = R$ for some $\lambda_1 \in M^*$.*

(5) *If $\lambda m + \pi = 1_S$, $\lambda \in M^*$, $m \in M$, $\theta^2 = \theta \in S$, then $(m + m_1\theta)M^* = R$ for some $m_1 \in M$.*

Proof. (1)⇒(2)⇒(3). These are clear by Theorem 26.

(3)⇒(4). Let $\lambda m + \theta = 1_S$ as in (4). Then $\lambda m = 1_S - \theta$ is an idempotent so we have $\lambda(m\lambda m) + \theta = 1_S$. But $m\lambda m$ is regular $\big[$in fact $(m\lambda m)\lambda(m\lambda m) = m\lambda m\big]$ so (3) and Lemma 1 show that $M(\lambda + \theta\lambda_1) = R$ for some $\lambda_1 \in M^*$. This proves (4).

(4)⇒(5). This is by Lemma 1.

(5)⇒(1). Suppose that $\lambda m + \theta = 1_S$ where $\lambda \in M^*$, $m \in M$, and $\theta \in S$; we must prove that $(m + m_1\theta)M^* = R$ for some $m_1 \in M$. As S is an exchange ring, [11, Theorem 2.1] asserts that

$$\text{there exists } \pi^2 = \pi \in S \quad \text{such that} \quad \pi \in S\theta \text{ and } 1_S - \pi \in S(1_S - \theta).$$

Then $1_S - \pi \in S\lambda m \subseteq M^*m$, say $1_S - \pi = \lambda_1 m$ for some $\lambda_1 \in M^*$. Thus $\lambda_1 m + \theta = 1_S$, so (4) implies that $(m + m_2\theta_1)M^* = R$ for some $m_2 \in M$. But $\theta_1 \in S\theta$, say $\theta_1 = \theta_2\theta$ with $\theta_2 \in S$, so $m_2\theta = (m_2\theta_2)\theta = m_1\theta$ where $m_1 \in M$. Thus $(m + m_1\theta)M^* = R$, as required. □

It is interesting to note that the hypothesis in Theorem 29 that M has the finite exchange property is used only in to prove (5)⇒(1). However if $R = \mathbb{Z}$ then ${}_RR$ satisfies Condition (5) but is not an exchange ring.

5. Other Properties

In this brief section we present three results showing that well known properties of a ring with stable range 1 have analogues for any regular R-stable module. In all cases the module property actually characterizes the regular R-stable modules. In a subsequent paper we will present several other results of this type. In particular we will extend some cancellation results for modules.

Our first result requires the following lemma which is implicit in [16]. We include the short proof for completeness.

Lemma 30. *Zelmanowitz*. *Let ${}_RM$ be a module, and let $m \in M$ be regular, say $(m\lambda)m = m$, $\lambda \in M^*$. Then there exists $e^2 = e \in R$ and an R-isomorphism $\sigma : Rm \to Re$ where $m\sigma = e$ and $e = m\lambda$. Moreover*:

$$M = Rm \oplus W \quad \textit{where} \quad W = \{w \in M \mid (w\lambda)m = 0\}.$$

Proof. Let m and λ be as given. Write $e = m\lambda$ so that $e^2 = e$. Moreover $(Rm)\lambda = Re$ so $\sigma = \lambda|_{Rm}$: $Rm \to Re$ is epic, and σ is monic because $(rm)\sigma = 0$ means $(rm)\lambda = 0$ so $rm = r[(m\lambda)m] = 0$. Hence σ satisfies our requirements because $m\sigma = e$. Finally, since $m_1 - (m_1\lambda)m \in W$ for every $m_1 \in M$, we have $M = Rm + W$; this is direct because $rm \in W$ means $0 = (rm)\lambda m = rm$. □

With this, we can characterize the regular R-stable modules by a kind of comparability. If K and M are modules write $K \lesssim^{\oplus} M$ if K is isomorphic to a direct summand of M.

Theorem 31. *The following are equivalent for a module* ${}_RM$:
(1) M *is regular* R*-stable.*
(2) *If* $R = A_1 \oplus L$ *and* $M = A_2 \oplus N$ *with* $A_1 \cong A_2$*, then* $L \lesssim^{\oplus} N$.

Proof. (1)⇒(2). Given the situation in (2), let $\theta : M \to M$ be the projection with $M\theta = A_2$ and $\ker(\theta) = N$, and let $\varphi : A_1 \to A_2$ be an R-isomorphism. Choose $e^2 = e \in R$ such that $A_1 = Re$ and $L = R(1-e)$. Then we have $M \xrightarrow{\theta} A_2 \xrightarrow{\varphi^{-1}} A_1$ so we define $\lambda = \theta\varphi^{-1} \in M^*$. Write $m = e\varphi \in A_2 = M\theta$. Then $m\theta = m$, so

$$m\lambda = (m\theta)\varphi^{-1} = m\varphi^{-1} = (e\varphi)\varphi^{-1} = e.$$

It follows that $(m\lambda)m = em = m$, so m is regular. But then m is unit regular by (1), so Lemma 23 gives $(m\gamma)m = m$ for some $\gamma \in M^*$ with $M\gamma = R$. Now apply Lemma 30 to conclude that $M = Rm \oplus W$ where $W = \{w \in M \mid (w\gamma)m = 0\}$. In particular, $W \cong M/Rm = M/A_2 \cong N$, so it remains to show that $L \lesssim^{\oplus} W$.

To this end, define $\eta : W \to L = R(1-e)$ by $w\eta = (w\gamma)(1-e)$ for all $w \in W$. We claim that η is epic. Observe first that $\gamma : M \to R$ is epic and so splits, say $\phi\gamma = 1_R$ where $\phi \in \hom(R, M)$. Write $w_o = (1-e)\phi$ so that $w_o\gamma = 1-e$. Moreover, since $em = m$ we have $w_o \in W$ because $(w_o\gamma)m = (1-e)m = 0$. But then $w_o\eta = (w_o\gamma)(1-e) = (1-e)^2 = 1-e$, so $(1-e) \in W\eta$. It follows that $W\eta = R(1-e)$, proving that $\eta : W \to L$ is epic, as claimed. This means that η splits (L is projective), say $W = \ker(\eta) \oplus B$ where $B \cong L$. Thus $L \lesssim^{\oplus} W$, completing the proof of (2).

(2)⇒(1). Let $m \in M$ be regular, say $(m\lambda)m = m$ where $\lambda \in M^*$; we must show that m is unit regular. By Lemma 30 write $M = Rm \oplus W$ where an isomorphism $\sigma : Rm \to Re$ exists with $m\sigma = e$ and $e = m\lambda$. Since $R = Re \oplus R(1-e)$, (2) implies that $R(1-e) \lesssim^{\oplus} W$. In particular there exists an R-epimorphism $\tau : W \to R(1-e)$. With this define

$$\gamma : M = Rm \oplus W \to R = Re \oplus R(1-e) \quad \text{by} \quad (rm + w)\gamma = (rm)\sigma + w\tau.$$

for all $r \in R$ and $w \in W$. Since τ and σ are epic it follows that $\gamma \in M^*$ is epic (and so left invertible). But $(m\gamma)m = (m\sigma)m = em = m$, so this shows that m is unit regular, proving (1). □

Our second result is related to the following notion. A principal left ideal Ra in a ring R is called **uniquely generated** if $Ra = Rb$, $b \in R$ implies that $a = ub$ for some unit u in R. In [8] Kaplansky asked when this holds, and the question remains open. In 2006, Marks [10] proved that a regular ring is unit regular if and only if every principal left ideal is uniquely generated. Since unit regular rings have stable range 1, the following is a module analogue.

Theorem 32. *Let* ${}_RM$ *be a projective left* R*-module. If* M *has the finite exchange property, then the following are equivalent:*
(1) M *is regular* R*-stable.*
(2) *For any* $m \in M$ *and* $\theta \in \text{end}(M)$*, if* $Rm = M\theta$ *then* $m = m_1\theta$ *for some right invertible* $m_1 \in M$.

Proof. (1)⇒(2). Suppose that $Rm = M\theta$ with $m \in M$ and $\theta \in \text{end}(M)$. Define $\alpha : R \to Rm$ by $r\alpha = rm$. Since M is projective and $Rm = M\theta$, we can find $\lambda \in M^*$ such that $\theta = \lambda\alpha$. For any

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow\theta & & \\ R & \xrightarrow{\alpha} & Rm & \to & 0 \end{array}$$

$x \in M$, we see that $x\theta = (x\lambda)\alpha = (x\lambda)m = z(\lambda m)$. This shows that $\theta = \lambda m$. Write $m = m_1\theta$ for some $m_1 \in M$. But M is stable by Theorem 29 so, since $\lambda m_1 + (1_M - \lambda m_1) = 1_M$, Lemma 1 shows that $\big(m_1 + m_2(1_M - \lambda m_1)\big)M^* = R$ for some $m_2 \in M$. Hence $u = m_1 + m_2(1_M - \lambda m_1)$ is right invertible. Clearly $\theta = \lambda m = \lambda m_1\theta$, so $u\theta = \big(m_1 + m_2(1_M - \lambda m_1)\big)\theta = m_1\theta + 0 = m$, as required.

(2)⇒(1). If $m \in M$ is regular we show that m is unit regular and invoke Theorem 29. Let $m = (m\lambda)m$, $\lambda \in M^*$, so $Rm = M(\lambda m)$ and $\lambda m \in \text{end}(M)$. By (2) there exists a right invertible $m_1 \in M$ such that $m = m_1(\lambda m)$. But λm is an idempotent in end(M), so this shows that m is unit regular by Lemma 23, as required. □

We need some notation to formulate our third result. Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context. If $f^2 = f \in S$ and $e^2 = e \in R$, the following are equivalent:

(1) $f = wv$ and $e = vw$ for some $v \in V$ and $w \in W$.

(2) $f = wv$ and $e = vw$ where $v \in V$, $w \in W$, $vwv = w$ and $wvw = w$.

In fact, (1)⇒(2) by passing to $w_1 = fwe$ and $v_1 = evf$.

Definition. *When the above conditions are satisfied, we write $f \approx e$.*

Theorem 33. *Assume that a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ satisfies the following condition:*

$$\text{Each } R\text{-epimorphism } \lambda : {}_RV \to R \text{ has the form } \lambda = \cdot w \text{ for some } w \in W. \qquad (\dagger)$$

Then the following conditions are equivalent:[4]

(1) The Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is regular R-stable.

(2) If $f \approx e$, $f = f^2 \in S$, $e = e^2 \in R$, then $fw_0 = w_0e$ for some left invertible $w_0 \in W$.

(3) If $f \approx e$, $f = f^2 \in S$, $e = e^2 \in R$, then $v_0f = ev_0$ for some right invertible $v_0 \in V$.

Proof. Throughout the proof $f \in S$ and $e \in R$ denote idempotents.

(1)⇒(2). Let $f \approx e$, say $f = wv$ and $e = vw$ where $vwv = v$ and $wvw = w$. In particular v is unit regular by (1) and Theorem 29, so Theorem 27 shows that

$$v = (vw_1)v = (vw_1)v_1 = v_1\,(w_1v) \quad \text{where } w_1 \in W \text{ and } v_1 \in V \text{ right invertible.}$$

Let $v_1w' = 1_R$ where $w' \in W$. Then these equations give

$$vw' = (vw_1v_1)w' = vw_1, \quad \text{and so} \quad vw'v = vw_1v = v, \quad \text{and} \quad fw'v = wvw'v = wv = f.$$

Now define $s = 1_S - f - w'v \in S$ and $r = 1_R - e - vw_1 \in R$, and observe that

$$vs = -vf = -vwv = -v \quad \text{and} \quad (vw_1)r = -vw_1e, \quad \text{so} \quad sw'v = -fw'v = -f.$$

Finally, define

$$w_0 = sw'r \in W.$$

Since $fw_1e = w(vw_1v)w = wvw = w$, we obtain:

$$fw_0 = w(vs)w'r = w(-v)w'r = (-w)(vw_1)r = (-w)(-vw_1e) = fw_1e = w;$$

[4] Condition (†) is only required for (3)⇒(1).

$w_0e = sw'(re) = sw'(-vw_1e) = -s(w'v)(w_1e) = f(w_1e) = w.$

In particular $fw_0 = w_0e$, so it remains to show that w_0 is left invertible. This follows if we can show that both s and r are invertible (then $Vw_0 = Vsw'r = Vw'r = Rr = R$). In fact, $s^2 = 1_S$ and $r^2 = 1_R$. To see this, first observe that $vw'v = v = vw_1v$ and we get $fw'v = f = fw_1v$ and $w'vf = w'v = w_1vf$. Hence:

$$\begin{aligned} s^2 &= (1_S - f)^2 - (1_S - f)w'v - w'v(1_S - f) + (w'v)^2 = 1_S, \\ r^2 &= (1_R - e)^2 - (1_R - e)w_1v - w_1v(1_R - e) + (w_1v)^2 = 1_R. \end{aligned}$$

This proves (2).

(2)⇒(3). Let $f \approx e$, say $f = wv$ and $e = vw$ where $vwv = v$ and $wvw = w$. Note that $vf = v = ev$ and $we = w = fw$. By (2), let $w_2 \in W$ be left invertible such that

$$fw_2 = w_2e \qquad \text{and let} \qquad v_2w_2 = 1_R, \;\; v_2 \in V.$$

With this define

$$v_1 = v + (1_R - e)v_2 \in V \qquad \text{and} \qquad w_1 = w + (1_S - f)w_2 \in W$$

Since $vf = v$ and $we = w$, we obtain

$$\begin{aligned} v_1w_1 &= vw + v(1_S - f)w_2 + (1_R - e)vw + (1_R - e)v_2(1_S - f)w_2 \\ &= e \quad + \quad 0 \quad + \quad (1_R - e)(vw)e + \quad (1_R - e) \\ &= 1_R + (1_R - e)(v_2w)e. \end{aligned}$$

Hence v_1w_1 is a unit in R, and so v_1 is also right invertible. Since $e(1_R - e) = 0$ we have:

$$\begin{aligned} (v_1w_1)v_1 &= [1_R + (1_R - e)v_2we][v + (1_R - e)v] \\ &= v + (1_R - e)v_2 + (1_R - e)v_2wv + 0 \\ &= v + (1_R - e)v_2(1_S - f) \end{aligned}$$

Now define $v_0 = (v_1w_1)v_1$. Then v_0 is right invertible (because v_1w_1 is a unit in R) and we have $v_0f = vf = v = ev = ev_0$. This proves (3).

(3)⇒(1). Let $v \in V$ be regular; we show that v is unit regular and invoke Theorem 25. Let $v = (vw)v$, $w \in W$, where we may assume that $wvw = w$. Write $f = wv \in S$ and $e = vw \in R$, so $vf = v$ and $ev = v$. Then $f \approx e$ so, by (3), there exists a right invertible $v_0 \in V$ such that $v_0f = ev_0$. As $v_0W = R$, write $v_0w_0 = 1_R$ for some $w_0 \in W$. Now define R-linear maps θ and γ :

$\theta : Vf \to Re$ where $(xf)\theta = (xf)w_0$ for all $x \in V$.

$\delta : V(1_S - f) \to R(1_R - e)$ where $[x(1_S - f)]\delta = x(1_S - f)w_0(1_R - e)$ for all $x \in V$.

Then θ is epic because $(vf)\theta = (vf)w = vw = e$. Now observe that $(v_0f)w_0 = (ev_0)w_0 = e$. This shows that δ is also epic because

$$[v_0(1_S - f)]\delta = v_0(1_S - f)w_0(1_R - e) = (v_0w_0 - v_0fw_0)(1_R - e) = (1_R - e)^2 = 1_R - e.$$

With this, define $\gamma : V = Vf \oplus V(1_S - f) \to R = Re \oplus R(1_R - e)$ as follows:

$$[xf + y(1_S - f)]\gamma = (xf)\theta + [y(1_S - f)]\delta \text{ for all } x, y \in V.$$

Then γ is R-linear and

$$v\gamma v = (vf)\gamma v = (vf\gamma)v = (vf\theta)v = (vfw)v = v(wv) = v.$$

Moreover, $\gamma \in V^*$ is R-epic because both θ and δ are epic so, by (†), $\gamma = \cdot w$ for some $w \in W$. Hence $vwv = (v\gamma)v = v$, and it remains to show that w is left invertible. As γ is epic let $v_2\gamma = 1_R$, $v_2 \in V$. Then $v_2w = v_2\gamma = 1_R$, as required. □

Let ${}_RM$ be a module, let $\theta^2 = \theta \in \text{end}(M)$, and let $e^2 = e \in R$. As above, we write $\theta \approx e$ if $\theta = \alpha m$ and $e = m\lambda$ for some $m \in M$ and $\lambda \in M^*$. Condition (†) in Theorem 33 is clearly satisfied in the standard context of a module. Hence we obtain:

Theorem 34. *Given ${}_RM$, write $S = \text{end}(M)$. The following conditions are equivalent*:
(1) *M is regular R-stable.*
(2) *If $\theta \approx e$, $\theta = \theta^2 \in S$, $e = e^2 \in R$, then $\theta\lambda = \lambda e$ for some left invertible $\lambda \in M^*$.*
(3) *If $\theta \approx e$, $\theta = \theta^2 \in S$, $e = e^2 \in R$, then $m\theta = em$ for some right invertible $m \in M$.*

Acknowledgements: The authors would like to thank the referee for encouraging us to make the paper easier to read by working in an arbitrary Morita context and labeling our concepts generically. These changes have improved the exposition. We would also like to thank Dinesh Khurana for several e-mail conversations about IC rings. The research of the first author was supported by the Natural Science Foundation of Zhejiang Province (Y6090404). The second author is grateful for the support of the International Cooperation and Exchange Division of Hangzhou Normal University during his visit.

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- Questions relating to Corollary 3 and the fact that every ring with stable range 1 is directly finite
  - Find necessary conditions in an R- stable context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ that $vw = 1_R$ implies $wv = 1_S$.
  - Is there a counterexample to the converse of Proposition 4?
  - How does this relate to the problem "R directly finite implies $M_2(R)$ is directly finite?" Can we characterize when in context terms?
  - (Ken Goodearl at the Ohio meeting) If M is stable, when does the context ring $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ have stable range 1? True if the context is regular stable? If and only if?

- Questions relating to stable modules.
  - If M is a stable module, must end(M) have stable range 1?
  - If end(M) has stable range 1, must M be stable.

- Questions relating to the monoid of stable elements in R.
  - Let S denote the monoid of stable elements of R. If S is closed under sums, does R have stable range 1? What if S is a left ideal?
  - If every regular element of R is stable, is R an IC ring? (See Theorem **??**.)

- Questions relating to Theorem 6 and its Corollaries.
  - Every R- stable context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ takes the form $\begin{bmatrix} R & v_0 S \\ S w_0 & S \end{bmatrix}$ where $v_0 w_0 = 1_R$, and $(v_0 S)(S w_0) = R$ by Corollary 8. But $(S w_0)(v_0 S) = S f S$ so this shows that R and SfS are "Morita equivalent" except that SfS may not have a unity. Should we expand on this here? I think not.

- Questions relating to Theorem 16. (See referee report, paragraph 2): Can we use Theorem **??** to show:
  - Every unit regular ring is an IC-ring?
  - Every one-sided artinian ring is an IC-ring?
  - Every CS ring is an IC-ring is an IC-ring?
  - Every finite von Neumann algebra is an IC-ring?
  - If R is IC so also is eRe where $e^2 = e \in R$?

-