MATHEMATICS 271 L01 FALL 2003 QUIZ 3 SOLUTION

1. Let $\mathcal{R}$ be a relation on $\mathbb{N}\times\mathbb{N}$ defined by: for any $(a,b),(c,d)\in\mathbb{N}\times\mathbb{N}$, $(a,b)\,\mathcal{R}\,(c,d)$ if and only if $a-d=c-b$.
(a) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}$.
(b) List all elements of the equivalence class $[(3,4)]$.
(c) Find an equivalence class which has exactly 271 elements.
Solution: (a) We show that $\mathcal{R}$ is reflexive, symmetric and transitive on $\mathbb{N}\times\mathbb{N}$.

$\mathcal{R}$ is **reflexive:** Let $(a,b)\in\mathbb{N}\times\mathbb{N}$. Since $a-b=a-b$, we have $(a,b)\,\mathcal{R}\,(a,b)$. So, $\mathcal{R}$ is reflexive.

$\mathcal{R}$ is **symmetric:** Let $(a,b),(c,d)\in\mathbb{N}\times\mathbb{N}$, and suppose that $(a,b)\,\mathcal{R}\,(c,d)$; that is, $a-d=c-b$ and so $c-b=a-d$, which means $(c,d)\,\mathcal{R}\,(a,b)$. Thus, $\mathcal{R}$ is symmetric.

$\mathcal{R}$ is **transitive:** Let $(a,b),(c,d),(e,f)\in\mathbb{N}\times\mathbb{N}$, and suppose that $(a,b)\,\mathcal{R}\,(c,d)$ and $(c,d)\,\mathcal{R}\,(e,f)$; that is, $a-d=c-b$ and $c-f=e-d$. It follows that $a+b=c+d$ and $c+d=e+f$. Thus, $a+b=e+f$ and so $a-f=e-b$ which implies that $(a,b)\,\mathcal{R}\,(e,f)$. Thus, $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, symmetric and transitive on $\mathbb{N}\times\mathbb{N}$, $\mathcal{R}$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}$.
(b) The elements of the equivalence class $[(3,4)]$ are:
$(0,7),(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$ and $(7,0)$.
(c) The equivalence class $[(0,270)]$ has exactly 271 elements, which are $(0,270),(1,269),\ldots,(270,0)$.

2. An urn contains ten white balls numbered from 1 to 10 and ten black balls numbered from 1 to 10. A sample of four balls is selected. For each of the following questions, give a brief explanation on how you got the answer and simplify your answers.
(a) How many possible samples are there in total?
(b) How many samples are there in which there is at least a white ball?
(c) How many samples are there in which there are more white balls than black balls?
(d) How many samples are there in which the numbers on the balls are distinct?
Solution: (a) The answer is $\binom{20}{4}=\frac{20\times19\times18\times17}{4\times3\times2\times1}=4845$. This is the numberof ways to choose 4 balls from 20 balls.
(b) The answer is $\binom{20}{4}-\binom{10}{4}=4845-210=4635$, which is the total number of all possible samples minus the number of samples in which there are no white balls. The number of samples in which there are no white balls is $\binom{10}{4}$ which is the number of ways to choose 4 black balls from 10 blask balls.
(c) The answer is $\binom{10}{4}+\binom{10}{3}\binom{10}{1}=210+120\times10=1410$. This is because either all four balls are white (there are $\binom{10}{4}$ of such samples by choosing 4 white balls from 10 white balls) or the number of white balls is 3 and the number of black balls is 1 (there are $\binom{10}{3}\binom{10}{1}$ of such samples, by first choose 3 white balls from 10 white balls and then choose 1 black ball from 10 black balls).
(d) The answer is $\binom{10}{4}\times2^4=210\times16=3360$. This is because we can choose 4 numbers from 10 numbers and for each of the four chosen number, we have two choices of putting it on a white ball or a black ball.