MIDTERM WINTER 99

[10] **1**. Define the relation $\mathcal{R}$ on the set $\mathbb{Q}$ of rational numbers by: for any $a, b \in \mathbb{Q}$, $(a, b) \in \mathcal{R}$ if and only if $a + b$ is an integer.
(a) Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive? Give reasons.
(b) Is $\mathcal{R}$ an equivalence relation? Explain.
(c) Is $\mathcal{R}$ a partial order? Explain.

[12] **2**. (a) Either prove or give a counterexample:

$$\text{for all sets } A, B, C, \text{if } A \subseteq B \cup C \text{ and } B \subseteq A \cup C, \text{ then } A = B.$$

(b) Write out the *contrapositive* of the statement in part (a). Is it true or false? Explain.
(c) Write out the *converse* of the statement in part (a). Is it true or false? Explain.

[7] **3**. Let $\mathcal{F}$ be the set of all functions from $\mathbf{N}$ to $\mathbf{N}$, where $\mathbf{N}$ is the set of all positive natural numbers. Define a relation $\sim$ on $\mathcal{F}$ by: $f \sim g$ if $f(5) = g(5)$. Define functions f and g in $\mathcal{F}$ by:

$$f(x) = x + 3 \quad \text{and} \quad g(x) = 2x - 1.$$

(a) Is $f \sim g$? Explain.
(b) Is $f \circ g \sim g \circ f$? Explain.
(c) Find a function $h \in \mathcal{F}$ such that $f \circ h \sim g \circ h$.

[6] **4**. Prove by induction that, for all positive integers n,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1}$$

MIDTERM WINTER 98

[12] **1**. Let $\mathcal{A} = \{-3, 0, 2, 3, 4\}$,and define a relation $\mathcal{R}$ on $\mathcal{A}$ as follows:

$$\text{for all } x, y \in \mathcal{A},\ (x, y) \in \mathcal{R} \text{ if and only if } x \mid y.$$

(a) Draw the directed graph of $\mathcal{R}$.
(b) Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive? Provide a brief explanation to justify your answers.

[6] **2**. Prove that for all sets A, B and C, $(A \cap B) - C \subseteq A - (B \cap C)$.
(A Venn diagram does not constitute a proof.)

[8] **3**. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$\text{for all integers } x \in \mathbb{Z},\ f(x) = 9x^2 + 6x + 1.$$

(a) Is f one-to-one? Explain.
(b) Is f onto? Explain.

[13] **4**. *For this question, use only the definitions of rational and irrational numbers; do not use any other results about rational and irrational numbers, other than the fact that* $\sqrt{2}$ *is irrational.*
Let P be the statement: "For all real numbers a and b, if a is rational and b is irrational, then $a - b$ is irrational."
(a) Is P true or false? Give a proof or a counterexample.
(b) Write out (in good English) the *converse* of P. Is it true or false? Explain.
(c) Write out (in good English) the *contrapositive* of P. Is it true or false? Explain.

[11] **5**. Let the sequence $a_1, a_2, a_3, \cdots$ be defined as follows: $a_1 = 1$, and for all integers $n \geq 2$, $a_n = 2a_{n-1} + 5$. Prove that $a_n + 1$ is divisible by 4 for all integers $n \geq 2$.

MIDTERM FALL 99.

[7] **1.** Prove by induction that:

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1} \text{ for all integers } n\geq 1.$$

[10] **2**. For each of the following statement, determine whether the statement is true or false and prove your answer.

(a) For all sets A, B, C and D, $(A-B)\times(C-D)\subseteq(A\times C)-(B\times D)$.

(b) For all sets A, B, C and D, $(A-B)\times(C-D)\supseteq(A\times C)-(B\times D)$.

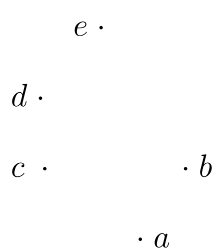
(c) For all sets A, B, C and D, $(A-B)\times(C-D)=(A\times C)-(B\times D)$.

[10] **3**. Let A be a subset of $\mathbb{N}$. Let $\mathcal{R}$ be the relation on A defined by $(a, b)\in\mathcal{R}$ if and only if $b=ka$ for some integer k.

(a) Prove that $\mathcal{R}$ is a partial on A.

(b) Let $A=\{1,2,3,4,5,6\}$. Draw the Hasse diagram of $(A,\mathcal{R})$?

(c) Let $A=\{a, b, c, d, e\}\subseteq\mathbb{N}$. Find some values of a, b, c, d and e so that the Hasse diagram of $(A,\mathcal{R})$ is below



[8] **4**. Define $f:\mathbb{Z}\to\mathbb{Z}$ by $f(x)=x^3+x$ for all $x\in\mathbb{Z}$.

(a) Is f one-to-one? Prove your answer.

(b) Is f onto? Prove your answer.

MIDTERM SPRING 99

[6] **1.** Find $\gcd(248, 172)$ using Euclidean Algorithm and find some integers x and y so that $\gcd(248, 172) = 248x + 172y$.

[10] **2**. Let $\mathcal{P}$ be the statement:

"For all sets $A,\ B$ and C, if $A \cap C = \phi$ then $(A \cup B) - C = A \cup (B - C)$."

(a) Is $\mathcal{P}$ true? Give a proof or counterexample.
(b) State the *converse* of $\mathcal{P}$. Is the *converse* of $\mathcal{P}$ true ? Explain.
(c) State the *contrapositive* of $\mathcal{P}$. Is the *contrapositive* of $\mathcal{P}$ true ? Explain.

[12] **3**. Let $A \subseteq \mathbb{N}$ and let $\mathcal{R}$ be the relation on A defined by:

for all a and $b \in A$,$(a, b) \in \mathcal{R}$ if and only if $a \mid b$.

(a) Show that $\mathcal{R}$ is a partial order on A.
(b) Let $A = \{1, 2, 3, 4, 5, 6\}$. Draw the Hasse diagram of $(A, \mathcal{R})$.
(c) With $A = \{1, 2, 3, 4, 5, 6\}$, what are the maximal element(s) of $(A, \mathcal{R})$? What are the minimal element(s) of $(A, \mathcal{R})$?

[12] **4**. For each of the following statements, determine whether it is true or false and prove your answer.
(a) For all functions $f : A \to B$ and $g : B \to C$, if both f and g are onto then $g \circ f$ is onto.
(b) For all functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is onto then f is onto.
(c) For all functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is onto then g is onto.

[10] **5**. The sequence $a_0, a_1, a_2, ...$is defined by $a_0 = 12,\ a_1 = 29$ and $a_k = 5a_{k-1} - 6a_{k-2}$ for all integers $k \geq 2$.
(a) Calculate a_2, a_3, a_4 and a_5.
(b) Prove by induction that $a_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 2$.