FINAL EXAMINATION
MATH271 (L 60)

SUMMER 2003 Time: 3 hours

[10] 1. In this question, the three parts are related.

(a) Find $\gcd(844, 375)$ and find some integers x and y so that $\gcd(844, 375) = 844x + 375y$.

(b) Is 375 invertible in Z_{844}? Explain. If 375 is invertible in Z_{844}, find the inverse (reciprocal) of 375 in Z_{844}.

(c) In Z_{844}, solve the equation $375 \otimes x = 2$.

[10] 2. Let $\mathcal{P}$ be the statement: "*If* $A \subseteq B \cup C$ *and* $B \subseteq C \cup A$ *then* $A \triangle B = C$."

(a) Is $\mathcal{P}$ true for all sets A, B and C? Prove your answer.

(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true for all sets A, B and C? Prove your answer.

(c) Write the contrapositive of $\mathcal{P}$. Is the contrapositive of $\mathcal{P}$ true for all sets A, B and C? Prove your answer.

[10] 3. Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ (where $\mathbb{Z}$ is the set of all integers) defined by:
for all $a, b \in \mathbb{Z}$, $a\mathcal{R}b$ if and only if $5 \mid (2a + 3b)$

(a) Show that $\mathcal{R}$ is an equivalence relation on **Z.**

(b) Find five elements in the equivalence class of 3.

(c) How many equivalence classes are there?

[10] 4. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of all integers) be functions defined by $f(x) = 2x + 1$ and $g(x) = \left\lfloor \frac{x}{2} \right\rfloor$

(a) Is $g \circ f$ onto? Prove your answer.

(b) Is $g \circ f$ one-to-one? Prove your answer.

(c) Is $f \circ g$ onto? Prove your answer.

(d) Is $f \circ g$ one-to-one? Prove your answer.

[10] 5. Let a, b and c be positive integers.

(a) Prove that if $c \mid a$ and $c \mid b$ then $c \mid (a + 2b)$ and $c \mid (2a + 3b)$.

(b) Prove that if $c \mid (a + 2b)$ and $c \mid (2a + 3b)$ then $c \mid a$ and $c \mid b$.

(c) Use (a) and (b) to show that $\gcd(a, b) = \gcd(a + 2b, 2a + 3b)$.

[10] 6. Let $n > 1$ be a natural number and let a and x be integers. Use the definition of congruence modulo n to prove or disprove the following:

(a) If $a \equiv x \pmod{n}$ then $(a + 1) \equiv (x + 1) \pmod{n}$.

(b) If $a \equiv x \pmod{n}$ then $a^2 \equiv x^2 \pmod{n}$.

(c) If $a \equiv x \pmod{n}$ then $a^2 \equiv x^2 \left(\bmod n^2\right)$.

[10] 7. A sequence $a_0, a_1, a_2, \cdots$ is defined by: $a_0 = 0$, $a_1 = 1$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for all integers $n \geq 2$.

(a) Calculate a_2, a_3, a_4 and a_5.

(b) Prove by induction that $a_n = 2^n - 1$ for all integers $n \geq 0$.

[10] 8. Six-digit numbers are to be formed using only the digits $1, 2, 3, 4, 5, 6, 7, 8$.

(a) How many such numbers can be formed if repetitions of the digits are allowed?

(b) How many numbers in (a) contain at least one 3 and at least one 5?

(c) How many numbers in (a) contain exactly one 3, two 5's and three 6's?

[10] 9. Let $X = \{1, 2, 3, \cdots, n\}$ and let $\mathcal{F}$ be the set of all functions from X to X. Choose an element f of $\mathcal{F}$ and assume that all elements of $\mathcal{F}$ are equally likely. Let A be the event that f is one-to-one and let B be the event that $f(1) = 2$.

(a) Find $P(A)$.

(b) Find $P(B)$.

(c) Find $P(A \cap B)$.

(d) Are the events A and B independent? Explain.

[10] 10. Let K_n be the complete graph with n vertices.

(a) How many edges does K_n have? Explain.

(b) How many induced subgraphs does K_n have? Explain.

(c) How many spanning subgraphs does K_n have? Explain.

End of Examination

FINAL EXAMINATION
MATH271 (L 01)

FALL 2001 Time: 3 hours

[10] 11. Let $\mathcal{P}$ be the statement: "*If* $A \triangle B \subseteq C$ *then* $A - B \subseteq C - B$."

(a) Is $\mathcal{P}$ true for all sets A, B and C? Prove your answer.

(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true for all sets A, B and C? Prove your answer.

[10] 12. Let $X \subseteq \mathbf{N}$ where $\mathbf{N}$ is the set of all natural numbers.

(a) Use the definition of $|$ to prove that $(X, |)$ is a poset.

(b) Draw the Hasse diagram of the poset $P = (\{2, 3, 4, 5, 6, 7, 8, 9\}, |)$.

(c) Find two different linear extensions of P in part (b).

[10] 13. Let $\mathcal{R}$ be a binary relation on $\mathbf{Z}$ ($\mathbf{Z}$ is the set of all integers) defined by:
for all $a, b \in \mathbf{Z}$, $a\mathcal{R}b$ if and only if $4 \mid 3a + b$.

(a) Show that $\mathcal{R}$ an equivalence relation on $\mathcal{P}(S)$.

(b) Find three positive and three negative elements of the equivalence class $[2]$.

[10] 14. Prove the following:

(a) If $f : A \to B$ and $g : B \to C$ are onto functions then $g \circ f$ is onto.

(b) There are functions $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ so that f is one-to-one and $g \circ f = id_{\mathbf{N}}$ but $g \neq f^{-1}$. (Note that $\mathbf{N}$ is the set of all natural numbers and $id_{\mathbf{N}}$ is the identity function from $\mathbf{N}$ to $\mathbf{N}$)

[10] 15. In this question, the three parts are related.

(a) Find $\gcd(1345, 237)$ and find two integers m and n so that $\gcd(1345, 237) = 1345m + 237n$.

(b) In $\mathbf{Z}_{1345}$, is 237 invertible? If it is, find the reciprocal of 237.

(c) Find all elements $x \in \mathbf{Z}_{1345}$ so that $237 \otimes x = 2$.

[10] 16. Prove the following statements:

(a) For all integers a, b and c, if a and b are relatively prime and $a \mid bc$ then $a \mid c$.

(b) For all integers a, b and c, if a and b are relatively prime and $a \mid c$ and $b \mid c$ then $ab \mid c$.

[10] 17. The sequence $b_0, b_1, b_2, \cdots$ is defined as follows: $b_0 = 1, b_1 = 3, b_2 = 5$, and for any integer $n \geq 3$, $b_n = 3b_{n-2} + 2b_{n-3}$.

(a) Find b_3, b_4, b_5 and b_6.

(b) Prove that $b_n < 2^{n+1}$ for all integers $n \geq 1$.

(c) Prove that $b_n = 2b_{n-1} + (-1)^{n-1}$ for all integers $n \geq 1$.

[10] 18. A three digit positive integer is selected at random and assume that all three digit positive integers are equally likely. Let E be the event that the chosen number is divisible by 4 and let F be the event that the chosen number is divisible by 6.

(a) Find $P(E)$. Simplify your answer.

(b) Find $P(F)$. Simplify your answer.

(c) Find $P(E \mid F)$. Simplify your answer.

(d) Are E and F independent? Explain.

[10] 19. Consider the graph H below:

(a) Is H bipartite? Explain.

(b) Is H Eulerian? Explain

(c) Find a maximal independence set of H.

(d) Find a maximal clique of H.

(e) Find a spanning tree of H.

[10] 20. Let G be a graph with at least six vertices. Note that $\varpi(G)$ is the clique number of G.

(a) Prove that $\varpi(G) \geq 3$ or $\varpi\left(\overline{G}\right) \geq 3$.

(b) Is it true that for all graphs G with five vertices, $\varpi(G) \geq 3$ or $\varpi\left(\overline{G}\right) \geq 3$? Prove your answer.

End of Examination