MATHEMATICS 271 L01 FALL 2003 QUIZ 4 SOLUTION

1. Let $d_0 = 2$, $d_1 = 5$ and for $n > 1$, let $d_n = 5d_{n-1} - 6d_{n-2}$. Prove by induction on n that $d_n = 2^n + 3^n$ for all integers $n \geq 0$.

Solution:

Base cases: $(n = 0, 1)$ $d_0 = 2 = 1 + 1 = 2^0 + 3^0$ and $d_1 = 5 = 2 + 3 = 2^1 + 3^1$. Thus, $d_n = 2^n + 3^n$ for the integers $n = 0$ and $n = 1$.

Induction step: Let $k \geq 2$ be an integer and suppose that

$$d_m = 2^m + 3^m \text{ for all integers } m \text{ where } 0 \leq m < k. \qquad \text{(IH)}$$

We want to show that $d_k = 2^k + 3^k$.

We note that since $k \geq 2$, we have $0 \leq k - 1 < k$ and $0 \leq k - 2 < k$, and so by (IH), we have

$$d_{k-1} = 2^{k-1} + 3^{k-1} \text{ and } d_{k-2} = 2^{k-2} + 3^{k-2}. \qquad (1)$$

Now, since $k \geq 2$, from the definition of the sequence d_n, we have

$$\begin{aligned} d_k &= 5d_{k-1} - 6d_{k-2} \\ &= 5\left(2^{k-1} + 3^{k-1}\right) - 6\left(2^{k-2} + 3^{k-2}\right) && \text{by (1)} \\ &= 5 \times 2^{k-1} + 5 \times 3^{k-1} - 6 \times 2^{k-2} - 6 \times 3^{k-2} \\ &= 5 \times 2^{k-1} + 5 \times 3^{k-1} - 3 \times 2^{k-1} - 2 \times 3^{k-1} \\ &= 2 \times 2^{k-1} + 3 \times 3^{k-1} \\ &= 2^k + 3^k. && \text{as wanted.} \end{aligned}$$

Thus, by the Principle of Mathematical Induction (Strong form), we have proved that $d_n = 2^n + 3^n$ for all integers $n \geq 0$.

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = 2x^2 + x + 7$ for all $x \in \mathbb{Z}$.

(a) Is f onto? Prove your answer.

Solution: f is not onto because for any $x \in \mathbb{Z}$, we have

$$\begin{aligned} f(x) &= 2x^2 + x + 7 = 2\left(x^2 + \tfrac{1}{2}x + \tfrac{7}{2}\right) \\ &= 2\left(x^2 + 2\tfrac{1}{4}x + \left(\tfrac{1}{4}\right)^2 - \left(\tfrac{1}{4}\right)^2 + \tfrac{7}{2}\right) \\ &= 2\left(\left(x + \tfrac{1}{4}\right)^2 + \tfrac{55}{16}\right) \geq \tfrac{55}{8} > 0. \end{aligned}$$

So, $f(x) \neq 0$ for all $x \in \mathbb{Z}$ and so f is not onto.

(b) Is f one-to-one? Prove your answer.

Solution: f is one-to-one and here is a proof. Let $a, b \in \mathbb{Z}$ so that $f(a) = f(b)$, that is, $2a^2 + a + 7 = 2b^2 + b + 7$ which implies that $2a^2 - 2b^2 + a - b = a$. It follows that

$$(a - b)\left[2(a + b) + 1\right] = 0. \qquad (1)$$

Since $a, b \in \mathbb{Z}$, $2(a + b) + 1$ is an odd integer and so $2(a + b) + 1 \neq 0$, which together with (1) inplies that $a - b = 0$, and therefore $a = b$. Thus, f is one-to-one.