MATHEMATICS 271 L01 FALL 2003 MIDTERM SOLUTION

1. Let $\mathcal{P}$ be the statement: "if $A = B \cup C$ then $B = A - C$."
(a) Is $\mathcal{P}$ true for all sets A, B and C? Prove your answer.
(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true for all sets A, B and C? Prove your answer.
Solution:
(a) $\mathcal{P}$ is not true for some sets A, B and C. For example, with $A = B = C = \{1\}$, we have $A = \{1\} = \{1\} \cup \{1\} = B \cup C$, but $B = \{1\} \neq \emptyset = \{1\} - \{1\} = A - C$.
(b) The converse of $\mathcal{P}$ is: "if $B = A - C$ then $A = B \cup C$."

The converse of $\mathcal{P}$ is not true for some sets A, B and C. For example, with $A = B = \emptyset$ and $C = \{1\}$, we have $B = \emptyset = \emptyset - C = A - C$, but $A = \emptyset \neq \{1\} = \emptyset \cup \{1\} = B \cup C$.

2. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let $\mathcal{R}$ be the relation on 2^A defined by: for all $X, Y \in 2^A$, $X \mathcal{R} Y$ if and only if $X \subseteq Y \cup \{2\}$.
(a) Is $\mathcal{R}$ reflexive? symmetric? antisymmetric? transitive? Prove your answers.
(b) Is $\mathcal{R}$ an equivalence relation on 2^A? Explain.
(c) Let $B = \{7, 8, 9\}$. How many elements X of 2^A are there so that $X \mathcal{R} B$?
Solution:
(a) $\mathcal{R}$ is reflexive because for any $X \in 2^A$, it is clear that $X \subseteq X \cup \{2\}$ and so $X \mathcal{R} X$.

$\mathcal{R}$ is not symmetric because $\emptyset \mathcal{R} \{1\}$ for $\emptyset \subseteq \{1\} \cup \{2\}$, but $(\{1\}, \emptyset) \notin \mathcal{R}$ because $\{1\} \nsubseteq \{2\} = \emptyset \cup \{2\}$.

$\mathcal{R}$ is not antisymmetric because $\{2\} \mathcal{R} \emptyset$ and $\emptyset \mathcal{R} \{2\}$ (because $\{2\} \subseteq \{2\} = \emptyset \cup \{2\}$ and $\emptyset \subseteq \{2\} = \{2\} \cup \{2\}$, but $\emptyset \neq \{2\}$.

$\mathcal{R}$ is transitive and here is a proof. Let $X, Y, Z \in 2^A$ so that $X \mathcal{R} Y$ and $Y \mathcal{R} Z$, that means, $X \subseteq Y \cup \{2\}$ and $Y \subseteq Z \cup \{2\}$. We show that $X \subseteq Z \cup \{2\}$. Let $x \in X$. Then from $x \in X$ and $X \subseteq Y \cup \{2\}$, we have $x \in Y$ or $x = 2$. In the case that $x \in Y$, we get $x \in Z \cup \{2\}$ because $Y \subseteq Z \cup \{2\}$. In the case that $x = 2$, it is clear that $x \in Z \cup \{2\}$. Thus, $X \subseteq Z \cup \{2\}$ and so $X \mathcal{R} Z$.
(b) $\mathcal{R}$ is not an equivalence relation on 2^A because it is not symmetric.
(c) The condition $X \mathcal{R} B$ is equivalent to $X \subseteq \{2, 7, 8, 9\}$, that is, X is a subset of $\{2, 7, 8, 9\}$. Since $\{2, 7, 8, 9\}$ has 4 elements, it has 2^4 such subsets.

3. Prove by induction that $1^3 + 2^3 + ... + n^3 = \dfrac{n^2 (n+1)^2}{4}$ for all integers $n \geq 1$.
Solution:
Base case: $(n = 1)$ $1^3 = 1 = \dfrac{4}{4} = \dfrac{1 \times 4}{4} = \dfrac{1^2 (1+1)^2}{4}$.
Induction Step: Let $k \geq 1$ be an integer and suppose that

$$1^3 + 2^3 + ... + k^3 = \frac{k^2 (k+1)^2}{4}. \qquad \text{(IH)}$$

We want to show that $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \dfrac{(k+1)^2 (k+2)^2}{4}$.
Now,

$$
\begin{aligned}
1^3+2^3+\ldots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3 && \text{by (IH)}\\
&= (k+1)^2\left[\frac{k^2}{4}+(k+1)\right]\\
&= (k+1)^2\left[\frac{k^2}{4}+\frac{4(k+1)}{4}\right]\\
&= (k+1)^2\,\frac{k^2+4k+4}{4}\\
&= (k+1)^2\,\frac{(k+2)^2}{4}\\
&= \frac{(k+1)^2(k+2)^2}{4} && \text{as wanted.}
\end{aligned}
$$

Thus, by the Principle of Mathematical Induction, we have shown that $1^3+2^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be functions defined by $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ and $g(x) = 2x+1$ for any $x \in \mathbb{Z}$.

(a) Is $g \circ f$ one-to-one? Prove your answer.

(b) Is $f \circ g$ onto? Prove your answer.

Solution:

(a) $g \circ f$ is not one-to-one.

Proof: We note that for any $x \in \mathbb{Z}$, $g \circ f(x) = g(f(x)) = g\left(\left\lfloor \frac{x}{2} \right\rfloor\right) = 2\left\lfloor \frac{x}{2} \right\rfloor + 1$, and so $g \circ f(0) = 2\left\lfloor \frac{0}{2} \right\rfloor + 1 = 1 = 2\left\lfloor \frac{1}{2} \right\rfloor + 1 = g \circ f(1)$ but $0 \neq 1$. Thus, $g \circ f$ is not one-to-one.

(b) $f \circ g$ is onto.

Proof: We note that for any $x \in \mathbb{Z}$, $f \circ g(x) = f(g(x)) = f(2x+1) = \left\lfloor \frac{2x+1}{2} \right\rfloor = \left\lfloor x + \frac{1}{2} \right\rfloor = x$ and so $f \circ g$ is clearly onto.